Bergen Lectures 2001

Random Matrix Theory

Lecture 3

Recall from Lecture 2 that the universality conjecture reduces to the verification of Plancherel-Rota asymptotics for orthogonal polynomials with respect to the measure $e^{-V(x)}dx$. For this lecture, as in Lecture 2, we restrict ourselves to the special case $V(x) = x^{2m}$, $m = 1, 2, \ldots$, — but more later.

Orthogonal polynomials can be computed via the RHP of Fokas, Its and Kitaev: Suppose Y is a 2×2 matrix-valued function satisfying

(1)
- $Y = Y^{(N)}(z)$ analytic in $\mathbb{C}\setminus\mathbb{R}$
- $Y_+(z) = Y_-(z)\begin{pmatrix} 1 & e^{-V(z)} \\ 0 & 1\end{pmatrix}$, $z \in \mathbb{R}$

  $$Y_\pm(z) = \lim_{\varepsilon\downarrow 0} Y(z \pm i\varepsilon)$$
- $Y(z)\begin{pmatrix} z^{-N} & 0 \\ 0 & z^{N}\end{pmatrix} \to I = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$ as $z \to \infty$

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Then

$$Y_{11}(z) = \pi_N(z)$$

where $\pi_N(z) = z^N + \cdots$ is the N^{th} orthogonal monomial for $e^{-V(z)}dz$, etc.

So the problem of the asymptotics of $\pi_N(z)$, and $P_N(z)$, reduces to controlling the solution of the above RHP as $N \to \infty$.

Now it turns out that it is useful to make a preliminary rescaling of the problem,

$$z \to N^{\frac{1}{2m}} z$$

Or more precisely, set

$$(2) \qquad S(z) = S^{(N)}(z) \equiv \begin{pmatrix} N^{-\frac{1}{2m}} & 0 \\ 0 & N^{\frac{1}{2m}} \end{pmatrix} Y(N^{\frac{1}{2m}} z)$$

Then $S(z) = S^{(N)}(z)$ solves ($V(zN^{\frac{1}{2m}}) = N\,V(z)$), the equivalent RHP

$$(3) \begin{cases} \bullet \quad S(z) \text{ analytic in } \mathbb{C} \setminus \mathbb{R} \\ \bullet \quad S_+(z) = S_-(z) \begin{pmatrix} 1 & e^{-NV(z)} \\ 0 & 1 \end{pmatrix} \quad \text{for } z \in \mathbb{R} \\ \bullet \quad S(z) \begin{pmatrix} z^{-N} & 0 \\ 0 & z^{N} \end{pmatrix} \to I \quad \text{as } z \to \infty . \end{cases}$$

This scaling can be motivated as follows: we have the eigenvalue distribution

$$\frac{1}{Z_N} e^{-\sum V(x_i)} \prod_{i<j}(x_i-x_j)^2 \, d^N x$$

$$= \frac{1}{Z_N} e^{-\left[\sum_{i\neq j}\log|x_i-x_j|^{-1} + \sum V(x_i)\right]} \, d^N x$$

After scaling $x \to N^{\frac{1}{2m}} x$, the distribution takes the form

$$\frac{1}{Z_N'} e^{-\left[\sum_{i\neq j}\log|x_i-x_j|^{-1} + N\sum V(x_i)\right]} \, d^N x$$

For any $x = (x_1, \dots, x_N)$ let

$$h_x = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}$$

denote the normalized counting measure at the x_i's. Then the above distribution takes the form

$$\frac{1}{Z_N'} e^{-N^2 \Phi^V(h_x)} \, d^N_x$$

where

in which the repulsive "forces" between the charges are balanced on the same scale by the external potent

We expect the main contribution to come from a distribution where Φ^V is minimal. More precisely we are lead to consider the following auxiliary variation problem

(4) $$E^V = \inf_{\mu \in \mathcal{P}} \Phi^V(\mu) = \inf_{\mu \in \mathcal{P}} \left[\iint \log|s-t|^{-1} d\mu(s)\, d\mu(t) + \int V(s)\, d\right.$$

where $\mathcal{P}$ is the space of probability distributions on $\mathbb{R}$.

It turns out that the infimum is achieved at a unique $\mu = \mu^V$, the equilibrium measure

$$E^V = \Phi^V(\mu^V)$$

Moreover μ^V has compact support.

Clearly $d\mu^V$ can be thought of as the describing the equilibrium distribution of (repelling Coulombic) charges constricted lie on the line imbedded in the plane in the presence of an external field V. But it (also) turns out to be connected to a rather extraordinary collectio[n] of problems in classical analysis. We descri[be] two of these connections which are relevant to the problem at hand:

(i) In the scaled unitary ensemble

$$\frac{1}{Z_N} e^{-N \operatorname{tr} V(M)} dM \qquad (V(x) = x^{2m})$$

(5) $$\lim_{N\to\infty} \frac{1}{N} \mathbb{E}xp\,(\#\text{ eigenvalues in } B) = \mu^V(B)$$

for any Borel set B. It is this relation that teaches us how to adjust our "microscopes" to see t[he] universal beh[avior]

(ii) For the scaled measure $e^{-NV(x)}\,dx$? $\lambda_N \sim N \cdot N^{-\frac{1}{2m}} =$ $\underline{\underline{m=1}}$ $\lambda_N = N$

let $x_1^{(N)} \geq \dots \geq x_N^{(N)}$ denote the zeros of the N^{th} orthogonal polynomial $P_N(x)$. Define

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the normalized counting measure

$$\delta^{(N)} = \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j^{(N)}}$$

Then as $N \to \infty$,

(6) $$\delta^{(N)} \longrightarrow \mu^V$$

ie $\lim_{N \to \infty} \int f(s) d\delta^N(s) = \int f(s) d\mu^V(s)$

for all continuous functions f decaying at $s = \pm\infty$.

We will return to the relation (6) later on.

The Euler - Lagrange equations for the equilibrium measure $d\mu^V$ have the form: $\exists !\, l \in \mathbb{R}$ (the Lagrange multiplier) st

(i) $\quad 2 \int \log |x-y|^{-1} d\mu^V(y) + V(x) \geq l \qquad \forall x \in \mathbb{R}$

(ii) $\quad 2 \int \log |x-y|^{-1} d\mu^V(y) + V(x) = l \qquad \forall x \in \operatorname{supp} d\mu^V$

Conversely if $d\tilde{\mu}$ is <u>any</u> probability measure with compact support satisfying (i)(ii) for some real # $\tilde{l}$,

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necessarily $\tilde{\mu} = \mu^V$ and $\tilde{\ell} = \ell$.

In the special case $V(x) = x^{2m}$ these variatio[nal]

conditions can be solved explicitly (they reduce to a

scalar RHP) and one finds

(7) $$d\mu^V(z) = \psi(z)\, dz$$

where $\psi(z)$ is supported on a finite interval $(-a,$

(8) $$\psi(z) = \frac{m}{\pi} (a^2 - z^2)^{\frac{1}{2}}\, h(z)\, \chi_{(-a,a)}(z)$$

where

(9) $$h(z) = z^{2m-2} + \sum_{j=0}^{m-1} \beta_j z^{2j} \quad , \beta_j > 0 ,$$

and

(10) $$a = \left(m \prod_{k=1}^{m} \left(\frac{2k-1}{2k}\right) \right)^{-\frac{1}{2m}}$$

Aside: For more general weights $e^{-NV(x)}dx$ where

- $V(x)$ is analytic in a nbhood of $\mathbb{R}$
- $\frac{V(x)}{\log|x|} \to \infty$ as $|x| \to \infty$

one can show using ~~essentially~~ a mean field [...]
calculation going back essentially to Johansson that

measure

$$d\mu^V(x) = \psi(x)\,dx$$

and is supported on a finite union of interva

$\psi(x) > 0$

This turns out to be critical information in analyzing the asymptotics of OP's in the gener case.

We now return to the RHP (3). The calculations that follow are motivated by work which Zhou & I did together with Venakides on the zero-dispersion problem for KdV where genuinel non-linear oscillations ($\mathrm{sn}(\alpha x + \beta)$ as opposed to $\sin(\alpha x + \beta)$) develop.

Our first task is to turn the RHP i a normalized RHP. Set

To this end set

$$(11) \qquad T(z) = T^{(N)}(z) = \begin{pmatrix} e^{N\tilde{\ell}/2} & 0 \\ 0 & e^{-N\tilde{\ell}/2} \end{pmatrix} S(z) \begin{pmatrix} e^{-Ng(z)} & 0 \\ 0 & e^{Ng(z)} \end{pmatrix} \begin{pmatrix} e^{-} \\ 0 \end{pmatrix}$$

Here $\tilde{\ell}$ is a constant, to be determined below, an

$g(z)$ is an as yet undetermined function with

the following properties

(i) $g(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$

(ii) $g(z) \sim \log z$ as $z \to \infty$

A simple, direct calculation shows that $T(z)$ solve

the following RHP:

(12)
- $T(z)$ analytic in $\mathbb{C} \setminus \mathbb{R}$
- $T_+(z) = T_-(z) \; v_T(z)$, $z \in \mathbb{R}$

where $v_T(z) = \begin{pmatrix} e^{N(g_-(z) - g_+(z))} & e^{N(g_-(z)+g_+(z) - V + \tilde{\ell}} \\ 0 & e^{N(g_+(z) - g_-(z))} \end{pmatrix}$

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The normalization condition $T(z) \to I$ follows as

$$\begin{pmatrix} e^{-Ng(z)} & 0 \\ 0 & e^{Ng(z)} \end{pmatrix} \sim \begin{pmatrix} z^{-N} & 0 \\ 0 & z^{N} \end{pmatrix} \quad \text{as } z \to \infty$$

Note that the RHP (12) is equivalent to the original RHP (1) in the sense that the solution of one of them immediately gives the solution of the other.

We now list the properties that $g(z)$ should have: the reason for these properties will become clear a little further on:

(a) Suppose that on some finite interval $(-\tilde{a}, \tilde{a})$

$$g_+(z) + g_-(z) - V(z) + \tilde{\ell} = 0$$

(b) $$g_+(z) + g_-(z) - V(z) + \tilde{\ell} < 0 \quad \text{for } |z| > \tilde{a}$$

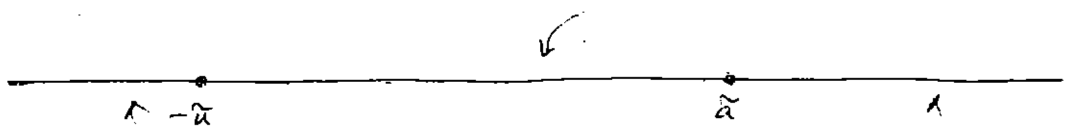
(c) $$\left.\begin{cases} g_+(z) - g_-(z) \in i\mathbb{R} \\ i\,\frac{\partial}{\partial z}\left(g_+(z) - g_-(z)\right) > 0 \end{cases}\right\} \quad \text{for } z \in (-\tilde{a}, \tilde{a})$$

(d) $$e^{N(g_+(z) - g_-(z))} = 1 \quad \text{for } |z| > \tilde{a}$$

Now observe that if g satisfies (a) and (d)

$v_T(z)$ takes the form

$$\begin{pmatrix} e^{N(g_- - g_+)} & 1 \\ 0 & e^{N(g_+ - g_-)} \end{pmatrix},$$



$$\begin{pmatrix} 1 & e^{N(g_+ + g_- - V + \tilde{l})} \\ 0 & 1 \end{pmatrix}$$

Set

$$G(z) = g_+(z) - g_-(z) \qquad \text{for} \qquad z \in (-\tilde{a}, \tilde{a})$$

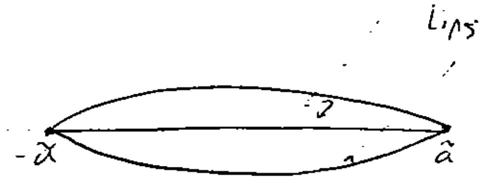
and observe that

$$\begin{pmatrix} e^{-NG} & 1 \\ 0 & e^{NG} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{NG} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-NG} & 1 \end{pmatrix}$$

Also as

$$\begin{aligned} G(z) &= g_+ - (V - \tilde{l} - g_+) = 2g_+ - V(z) + \tilde{l} \\ &= (V - \tilde{l} - g_-) - g_- = V(z) - \tilde{l} - 2g_- \end{aligned}$$

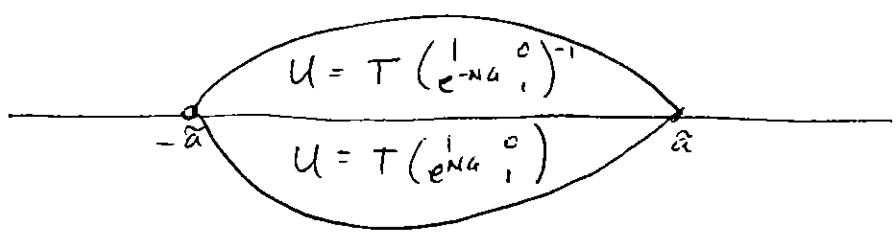
G has an analytic continuation to a lens



around $(-\tilde{a}, \tilde{a})$

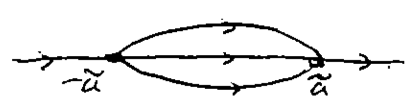
We now deform the RHP (12) for T by setting

$$U(z) = T(z)$$



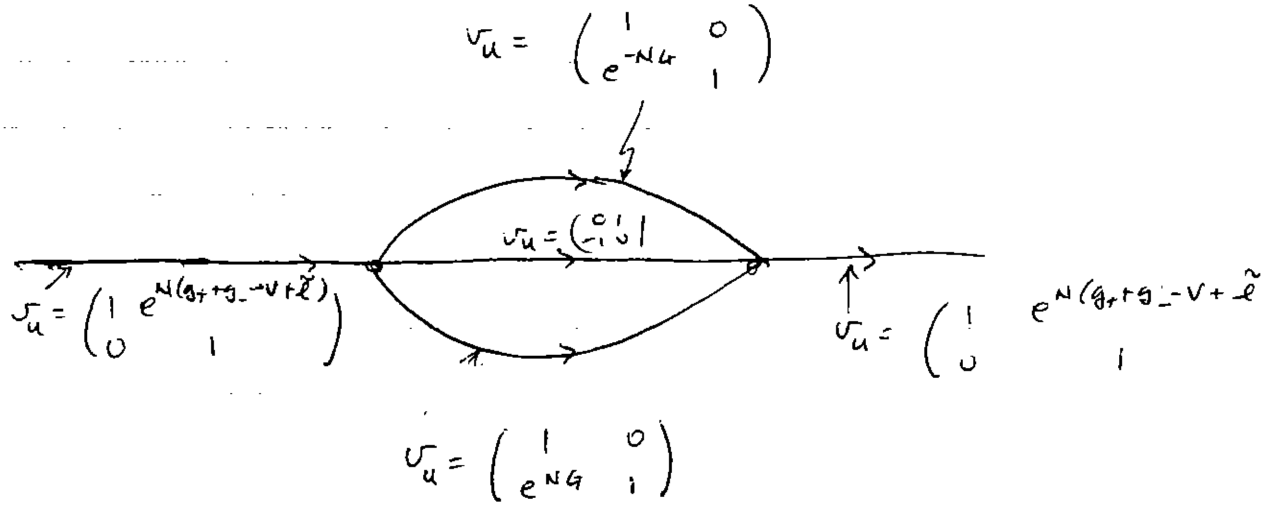
$$U(z) = T(z)$$

A simple calculation shows that U solves the equivalent RHP on the extended contour

$\Sigma_U =$ 

$$(13) \begin{cases} \bullet \quad U(z) \text{ anal. in } \mathbb{C} \setminus \Sigma_U \\ \bullet \quad U_+(z) = U_-(z) \, v_U(z) \end{cases}$$

where



Note finally that by (c) and the Cauchy Riemann conditions, $\operatorname{Re} G > 0$ on the upper lip and $\operatorname{Re} G < 0$ on the lower lip. And also by (b), $g_+ + g_- - V + \tilde{\ell} < 0$ for $|z| > \tilde{a}$.

At this stage, morally, we are done! Inde we see that the choices (a)..(d) were made precisely so that the jump matrix for v_u converges to a limiting form $v_\infty = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on (

and we expect that

$$U \to U^{\infty}$$

where U^{∞} solves the ~~RHP~~ RHP

$$(14)\quad \begin{cases} \bullet \quad U^{\infty}(z) \text{ anal. in } \mathbb{C} \setminus [\hat{a}, \hat{a}] \\ \bullet \quad U^{\infty}_{+}(z) = U^{\infty}_{-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad z \in (\hat{a}, \hat{a}) \\ \bullet \quad U^{\infty}(z) \to I \qquad \text{as } z \to \infty . \end{cases}$$

This RHP is easily solved explicitly and hence gives, after we unravel all our transformations, the leading asymptotics for the orthogonal polynom as desired.

Now how do we know that a $g(z)$ satisfying (a) .. (d) exists? This is where one of the truly lucky things in the the

Suppose μ is a probability measure and set

(15) $$g(z) = \int \log(z-s)\, d\mu(s)$$

Then a direct calculation shows that

$g(z)$ solves (a)...(d) $\iff$ $d\mu$ satisfies the Euler Lagrange eqtns (i)(ii) p(6)
ie $d\mu = d\mu^V$, the eqm.

So once we have computed the eqm. measure, we are done, at least morally.

Now why only morally? It turns out tha

to show that the solution of a RHP converg

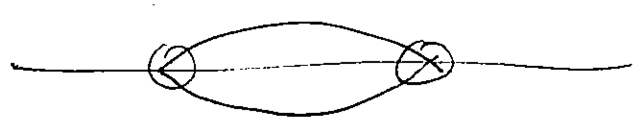
to the solution of some other RHP, we need

know that the jump matrices converge in th

$L^\infty(\Sigma)$ norm. But although

$v^{\infty}(z) \xrightarrow{\infty} v^{\infty}(z)$

pointwise on Σ_u, the rate of convergence is slower and slower near the endpoints $\pm \tilde{a}$, and so the convergence is not in L^∞. This is an absolutely major difficulty and most of the analytical work in the method goes into overcom. this difficulty. We proceed by constructing a local solutions to the RHP



and patching them all together to obtain a paramet. for the full problem.

The problem we face is very similar to the homogenizat. problem in elliptic pde theory eg suppose we have

$$H_\varepsilon = -\nabla \cdot a(\tfrac{x}{\varepsilon}) \nabla$$

so $H_\cdot$ converges weakly to some operator H ..

functions f e.g. $\frac{1}{H_z + 1}$ $\to$?

Finally we note that there is another way to see that $\begin{pmatrix} e^{-Ng(z)} & 0 \\ 0 & e^{Ng(z)} \end{pmatrix}$ is the "right" normalizer for the RHP. Recall that

$$Y_{11} = \pi_N(z) = \prod_{i=1}^{N} (z - x_i^{(N)})$$
$$= e^{\sum_i \log(z - x_i^N)}$$
$$= e^{N \int \log(z-s)\, d\delta^N(s)}$$

where again $\delta^N = \frac{1}{N} \sum_i \delta_{x_i^N}$ is the normalized counting measure for the zeros of π_N. But by

$$\delta^N \to d\mu^V$$

Thus we expect

$$Y_{11} \sim e^{N \int \log(z-s)\, d\mu^V(s)} = e^{Ng(z)}$$

Thus we anticipate that

$Y \begin{pmatrix} e^{-Ng} & 0 \end{pmatrix}$ $\left(Y_{11} e^{-Ng} \quad \right)$ is well