Bergen Lectures 2001

Random Matrix Theory

Lecture 4 In this the final lecture we consider Ulam's problem in combinatorics and show how 'integrable' methods lead to the solution. We will describe joint work with J. Baik & K. Johansson.

Let S_N be the group of permutations π of the #'s $1, \ldots$
For $1 \le i_1 < i_2 < \cdots < i_k \le N$ we say that $\pi(i_1), \ldots, \pi(i_k)$ is an increasing subsequence of length k in π if $\pi(i_1) < \cdots < \pi(i_k)$.
Let $l_N(\pi)$ be the length of the longest increasing subseq.

E.g. $N=5$ π : $\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 5 & 1 & 3 & 2 & 4 \end{matrix}$, 134 and 124 both length 3 & are the longest increasing subseq.

Equip S_N with the uniform distribution so that

$$q_{n,N}^{(1)} = \text{Prob}\{ l_N(\pi) \le n \}$$

$$= \frac{f_{n,N}}{N!}$$

$f_{n,N}$ = # of π's with $l_N(\pi) \le n$.

Aim: determine the asymptotics of $q_{n,N}$ as $N \to \infty$

Why is ~~this problem~~ l_N of interest?

Connections to

(i) representation theory of S_N. Young tableaux (more later)

(ii) $N - l_N(\pi)$ = min # of insertion / deletion ~~moves~~ steps needed to go from the identity (a library problem).

$\Rightarrow$ $d(\pi, \sigma) = N - l_N(\pi\sigma^{-1})$ Ulam's metric on S

Useful for a variety of statistical questions

(iii) DNA sequencing

(iv) l_N believed to lie a particular universality class for some statistical mechanical particle models. (first passage percolation questions): ~~random growth~~ growth models, exclusion process model, vicious walker models, superconduction

(v) patience sorting: algorithm

(vi) (random) topologies on surfaces (vi) physical ~~grow~~ problem: burning,

Some ~~h~~istory

The beginning of this business.

(1935) Erdos + Szekeres: proved a Ramsey theory type result: (from the book[!!]) every permutation of N #'s has either an inc. at least

~~Two~~ Independent proofs ~~law~~ of Ulam's problem have now b
given by

1995 Aldous & Diaconis

1996 Seppäläinen

1997 Johansson

⋮

Over the years many conjectures have been made about other statistics for ℓ_N. ~~Eg~~ In particular there were various conjectures for $\mathrm{Var}(\ell_N)$ of the form

$$\mathrm{Var}(\ell_N) \sim c N^{\alpha} \qquad ; \ \alpha = \tfrac{1}{3}, \ \text{Kesten}$$

for different values of α.

Then in typical fashion, together with Eric Rains Andrew Odlyzko in 1993 ~~the early~~ began a series of large scale ($N \sim 10^{[illegible]}$) Monte Carlo simulations on $\mathrm{Var}(\ell_N)$. They found that

$$\lim_{N\to\infty} \frac{\mathrm{Var}(\ell_N)}{N^{1/3}} = c_0 \sim 0.81[illegible]3$$

$$\lim_{N\to\infty} \frac{\mathbb{E}(\ell_N) - 2\sqrt{N}}{N^{1/6}} = c_1 \sim -1.[illegible]771$$

And that is where the matter stood

(

In order to state our results on the prob

I need to introduce ~~some notation~~ the Tracy Widom distribution. Let ~~that~~

$u(x)$ be the (unique) solution of the Painlevé (P II) equation

$$u_{xx} = 2u^3 + xu \quad , \quad u \sim -\underbrace{Ai(x)}_{\text{Airy function}} \text{ as}$$

(Hastings & McLeod: DIFT Its & Novok)

We have (& I will say more about this later)

$$u(x) = -Ai(x) + O\left(\frac{e^{-(4/3)x^{3/2}}}{x^{1/4}}\right), \quad x \to$$

$$= -\sqrt{\tfrac{-x}{2}}\left(1 + O\left(\tfrac{1}{x^2}\right)\right) \quad \text{as } x \to -$$

Set

$$F'(t) = e^{-\int_t^\infty (x-t)\, u^2(x)\, dx} \qquad \text{— TW idom}$$

$$\sim e^{+ct^3} \text{ as } t \to -\infty \qquad \sim 1 - e^{-t^{3/2}} \text{ as } t \to +\infty.$$

Thus $\frac{d}{dt}F^{(1)}(t) > 0$, $F^{(1)}(t) \to 0$ $t \to -\infty$, $F^{(1)}(t) \to 1$ as $t \to$

so $F^{(1)}(t)$ is a distribution function.

<u>Theorem 1</u> Let S_N be the group of perms of $1, \dots, N$ with uni dist$_n$ and let $l_N(\pi)$ be the length of the longest increasing subseq. for π. Let $\chi^{(1)}$ be a random variable whose distr. fn is $F^{(1)}$. Then as $N \to \infty$

$$\chi_N = \frac{l_N - 2\sqrt{N}}{N^{1/6}} \longrightarrow \chi \quad \text{in distrib.}$$

i.e.

$$\lim_{N\to\infty} \text{Prob}\left(\chi_N \le t \right) = F(t) \qquad \forall\, t \in \mathbb{R}$$

We also have convergence ~~of~~ ~~...~~ of the moments.

<u>Thm 2</u> For any $m = 1, 2, 3, \ldots$, we have

$$\lim_{N\to\infty} \mathbb{E}_N\left(\chi_N^m \right) = \mathbb{E}\left(\chi^m \right)$$

where $\mathbb{E}$ denotes exp. ~~...~~ wrt F. In particular for

$$\lim_{N\to\infty} \frac{\text{Var}(l_N)}{N^{1/3}} = \int t^2\, dF(t) - \left(\int t\, dF(t) \right)^2$$

and for $m=1$

$$\lim_{N\to\infty} \frac{\mathbb{E}_N(l_N) - 2\sqrt{N}}{N^{1/6}} = \int t\, dF(t)$$

If one solves PII numerically & computes the RHS

(a) (b) resp one finds 0.8132 and −1.771 which

pretty well the values (Monte Carlo)

... with ... & ... above ... ~~...~~ ~~...~~ ...

Now it turns out ~~that~~ there is a very interesting connection between the above results and another subject in a very different area, random matrices. In this theory (see [Mehta]) one considers in particular $N \times N$ Hermitian matrices $M = (M_{ij})$ with prob. density (GUE)

$$Z_N^{-1} e^{-\operatorname{tr} M^2} dM = \underset{\substack{\uparrow \\ \text{norm. const}}}{Z_N^{-1}} e^{-\operatorname{tr} M^2} \prod_{1}^{N} dM_{ii} \prod_{i<j} d(\operatorname{Re} M_{ij})\, d(\operatorname{Im} M_{ij})$$

Now the fact is this (Tracy – Widom) : as $N \to \infty$, the distribution of the largest eigenvalue λ_1 of a matrix M, suitably (centered & scaled $(\lambda_1 \to (\lambda_1 - \sqrt{2N}) N^{\frac{1}{6}} 2^{\frac{1}{2}})$), converges

i.e. $\operatorname{Prob}\left\{ \frac{\lambda_1 - \sqrt{2N}}{N^{-1/6} 2^{-1/2}} \le t \right\} \xrightarrow[N\to\infty]{} F(t)$

~~in distribution~~ precisely to the same function $F(t)$!

So our theorem says that ~~they~~ the length of the longest increasing subsequence of π behaves like the largest eigenvalue of a random matrix.

~~Natural Question:~~

Tracy and Widom also computed the distribution function for 2^{nd}, 3^{rd}, ... largest eigenvalues.

Natural Question: Is there anything in the rand permutation picture that behaves like the 2^{nd} (or 3^{rd}, 4^{th}..) largest eigenvalues of a random GUE ma

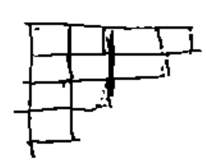
~~To answer this question~~

It ~~clearly~~ cannot be the 2^{nd} longest inc. subseq, which is clearly distributed in the same way as the longest.

To see what ~~to~~ to do we must introduce some st ideas from combinatorics / representation theory.

Let $\mu = (\mu_1, \ldots, \mu_\ell)$, $\mu_1 \geq \cdots \geq \mu_\ell \geq 1$ be a partition of N
$\sum_1^\ell \mu_i = N$

Associate to μ is a frame
Ferrer's diagram
Young diagram



$\mu_1 = 4$
$\mu_2 = 3$
$\mu_3 = 2$
$\mu_4 = 1$
$N = 10$.

If we insert the #'s $1, \ldots, N$ (bijectively) into the boxes of the the frame we obtain $N!$ Young ~~diagram~~ tableaux

$$\begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 6 & 3 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array} \qquad N = 6.$$

If we ensure that the rows and column are increasing

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$$

we obtain a Standard Young tableaux with ~~for~~ frame λ

Now there is a remarkable theorem of Robinson (1938) and Schensted (1961) ~~with~~ which says there is a bijection –

$\xrightarrow{RS}$

from S_N onto pairs of standard Young tableaux with the same

$$S_N \ni \pi \overset{R-S}{\longmapsto} \left(P(\pi), Q(\pi) \right)$$

the frame $(P(\pi))$ = frame $(Q(\pi)) = (\lambda(\pi))$.

Furthermore (Schensted '61) if

$$\ell_N(\pi) = \lambda_1(\pi)$$

[illegible]

tableaux behave statistically (with Plancherel measure = push forward unif dist. on S_N) like the ~~to~~ largest eig of a random GUE matrix.

What about $\lambda_2(\pi)$, the length of the 2nd row of boxes in the frame $(P(\pi))$ = frame $(Q(\pi))$?

Well if we go back to the simulation of Odlyzko and Rains, we find that ~~of~~ the computation indicate

$$\lim_{N\to\infty} \frac{\mathrm{Var}\,(\lambda_2(\pi))}{N^{1/3}} = .545$$

$$\lim_{N\to\infty} \mathbb{E}_N \left(\frac{\lambda_2(\pi) - 2\sqrt{N}}{N^{1/6}} \right) = -3.618$$

Well, ~~these~~ these values appear, once again, with the variance and mean of the 2nd largest eigenvalue ~~of~~ (suitably centered and scaled) of a random GUE matrix as computed by Tracy and Widom! ~~So the~~
We have the following result

$F(t) := e^{-\int_t^\infty (x-t) u^2(x;s)\,dx}$. $F^{(2)}(t) \equiv F(t) - \frac{\partial}{\partial s}\big|_{s=1} F(t,s)$ [illegible]

$F^{(1)}(t) = F(t,1)$. Let $\chi^{(2)}$ be a r.v. with distrib $F^{(2)}$

Thm 3 $\chi_n^{(2)} := (\lambda_2^{(n)} - 2\sqrt{n}) / n^{1/6} \Rightarrow \chi^{(2)}$

Thm 4 For $m = 0, 1, 2, \ldots$ $E_n((\chi_n^{(2)})^m) \to E((\chi^{(2)})^m)$

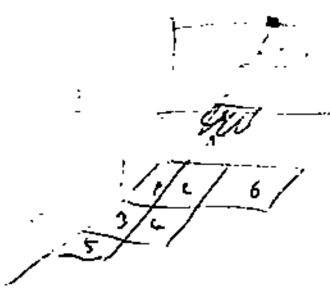
~~long open problem here is to~~ prove ~~the~~ the

analogs of Thm's 1 & 2 --- also --- for the 2^{nd} row, ~~open problem~~ conjectural: same for 3^{rd}, 4^{th} rows etc. Solved by Okounkov, Okounkov-Borodin, Johansson.

~~3^{rd} row, etc~~ Ultimately, we want to answer the

Q

following question: display $(P(\pi), Q(\pi))$ graphically



After centering $(-2\sqrt{n})$ and scaling $(\times \frac{1}{n^{1/6}})$ do the
5) ~~Forrester~~ Forrester (5) random walker models,
6) Baik, ~~?~~ Rains,
7) ~~Spohn~~ Spohn + Prahofer; growth models;
surfaces converge to some universal surfaces?

Explanation of results:
1) Tracy-Widom
2) Okounkov, Okounkov-Borodin
3) Johansson
4) Baik and Rains

First appearance of Random Matrix Th

Regev '81 $\sum_{\lambda \vdash n,\ \#\text{rows of } \lambda \le k} (f^\lambda)^\beta \underset{n\to\infty}{\sim} C_{n,k} \int \cdots \int dx_1 \cdots dx_k\, |D(x_1, \ldots x_n)|^\beta e^{-\Sigma \beta x}$ (explicit)

f^λ = # of standard tableaux of shape λ $D(x_1, \ldots, x_n) = \prod_{i<j} (x_i - x_j)$

How do we prove Theorems 1, 2, 3 and 4? The

key analytic fact is that the problems can

be rephrased as Riemann-Hilbert (RH) problems with (large) ~~exter~~ external parameters ~~(?)~~ This helps

for the following reason: in the '80s early ~~'80s~~ 90's

Xin Zhou and I introduced a steepest-descent type

method ~~for~~ to evaluate the asymptotic behavior of oscillatory ~~oscillating~~ RH problems, ~~with large~~

~~as some external param~~ This work was developed

by a # of people and eventually ('97) placed in a general form

by ~~(?)~~ Zhou, Venakides and myself. ~~in (?)~~ The

method is a non-commutative, nonlinear ~~(?)~~ in the case of RH problems of the classical

steepest descent method for saddle integrals. I'll say more

about this latter: also tomorrow's talk. There are now

many problems in many different cases that reduce to the asymptotic evaluation of some oscillatory RH

[illegible] Step 1 Poissonization

Set

$$\phi_n^{(1)}(\lambda) \equiv \sum_{N=0}^{\infty} \frac{e^{-\lambda}\lambda^N}{N!} q_{n,N}$$

By a rather elementary, but fortunate, de-Poissonization
story
(Tauberian Theorem: $q_{n,N}$ monotonic in N) due to Johansson,
lemma, the

asymptotics of $q_{n,N}$ as $N \to \infty$,
can be inferred from
The asymptotics of $\phi_n^{(1)}(\lambda)$ for $\lambda \sim N$

So we must investigate the double scaling limit for
$\Phi_n^{(1)}(\lambda)$, for $1 \le n \in N \sim \lambda$
critical region $n \sim 2\sqrt{\lambda}$ $\Leftarrow$

Poissonization helps because of the following wonderful fact

There is an exact formula for $\Phi_n^{(1)}(\lambda)$

$$\Phi_n^{(1)}(\lambda) = e^{-\lambda} D_{n-1}\left(e^{2\sqrt{\lambda}\cos\theta}\right) = e^{-\lambda} D_{n-1}(\lambda)$$

D_{n-1} is the $n \times n$ Toeplitz determinant with weight function
$f(e^{i\theta}) = e^{2\sqrt{\lambda}\cos\theta}$

$$D_{n-1}(f) = \det\left(c_{kj} = \int e^{-i(k-j)\theta} f(e^{i\theta}) \frac{d\theta}{2\pi} \right)$$
$0 \le k, j \le n$

This formula was first found by Gessel (1990) but has
since been discovered independently by many authors
Johansson, Diaconis – Shahshahani, Gessel – Weinstein – Wilf

Step 2

Set $\quad K_n^2(\lambda) = \dfrac{D_{n-1}(\lambda)}{D_n(\lambda)}$

$K_n^2(\lambda)$ is the normalization coefficient for the n^{th} orthonormal polyno. $\quad p_n(z) = K_n(\lambda)\,(z^n + \cdots$

wrt $\quad f(e^{i\theta})\frac{d\theta}{2\pi}$, $\quad \int \overline{p_n(e^{i\theta})}\, p_m(e^{i\theta})\, f(e^{i\theta}) \frac{d\theta}{2\pi} = \delta_{nm}$

$f = e^{i\lambda\cos\theta}$

Using the Strong Szegö limit theorem we have

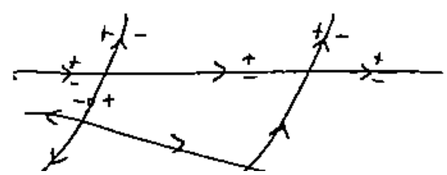
$$\log \Phi_n(\lambda) = \sum_{k=n}^{\infty} \log K_k^2(\lambda)$$

This is the main formula: to control $\Phi_n(\lambda)$ we must control $K_k(\lambda)$ for $(k \geq n, \lambda) \to \infty$

Step 3 Riemann-Hilbert problem

What is a RHP?

Suppose we have an oriented contour Σ in $\mathbb{C}$



Σ

By convention, if we traverse an arc in Σ in the direction of the arrow, we say that the $+$ (resp $-$) side lies to the left (resp. right)

Suppose we are given a (smooth) map $v : \Sigma \to$ Then the RH problem (Σ, v) is the following: find $m = m(z)$ st

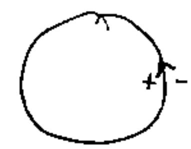
- m is analytic in $\mathbb{C} \setminus \Sigma$
- $m_+(z) = m_-(z)\, v(z)$, $z \in \Sigma$,

where $m_\pm(z)$ denote $\lim\limits_{\substack{z' \to z \\ z' \in \pm \text{ side}}} m(z')$.

If in addition

- $m(z) \to I$ we say that the RHP (Σ, v) is <u>normalized at ∞</u>.

For us the fact of the matter is the following (~~Fokas~~ following Fokas Its Kitaev): Let Σ = unit circle oriented counterclockwise:



Let $Y(z; k+1, \lambda)$ be the 2×2 matrix function satisfying the following RHP:

- $Y(z; k+1, \lambda)$ is analytic in $\mathbb{C}\setminus\Sigma$
- $Y_+(z; k+1, \lambda) = Y_-(z; k+1, \lambda) \begin{pmatrix} 1 & \frac{1}{z^{k+1}} e^{\sqrt{\lambda}(z+z^{-1})} \\ 0 & 1 \end{pmatrix}$
- $Y(z; k+1, \lambda) \begin{pmatrix} z^{-(k+1)} & 0 \\ 0 & z^{k+1} \end{pmatrix} = I + O\left(\frac{1}{z}\right)$ as $z \to \infty$

Then Y is unique and

$$\kappa_k^2(\lambda) = -Y_{21}(z=0; k+1, \lambda)$$

Also $p_{k+1}/\kappa_{k+1} = z^{k+1} + \cdots = Y_{11}$

So to evaluate $\kappa_k^2(\lambda)$, $k \geq n$, and hence $\phi_n^{(1)}(\lambda)$,

we must be able to control the solution of Y

the above RHP in the limit when the 2 "oscillating" parameters ~~[illegible]~~ k, $\sqrt{\lambda}$ in $z^{-(k+1)} e^{\sqrt{\lambda}}$ are large.

This is precisely the situation that can be controlled by the non-linear steepest descent method. The calculations are very similar to those that arose in the work of D. McLaughlin, Kriecherbauer, Venakides & Zhou on the proof of universality of various statistical quantities in random matrix theory.

<u>~~Stepanov Painleve~~</u>

<u>Remark</u> The above RHP for Y is ~~closely~~ equivalent related to the RHP's that arose in the analysis of the Toda lattice equations and the Painleve III equation. The fact that these

That $k_n^i(x)$, (and hence $q_{n,N}$) must satisfy some identities, what they are, we do not yet know, but this seems to be a ~~new~~ way to get new identities for relevant combinatorial quantities.

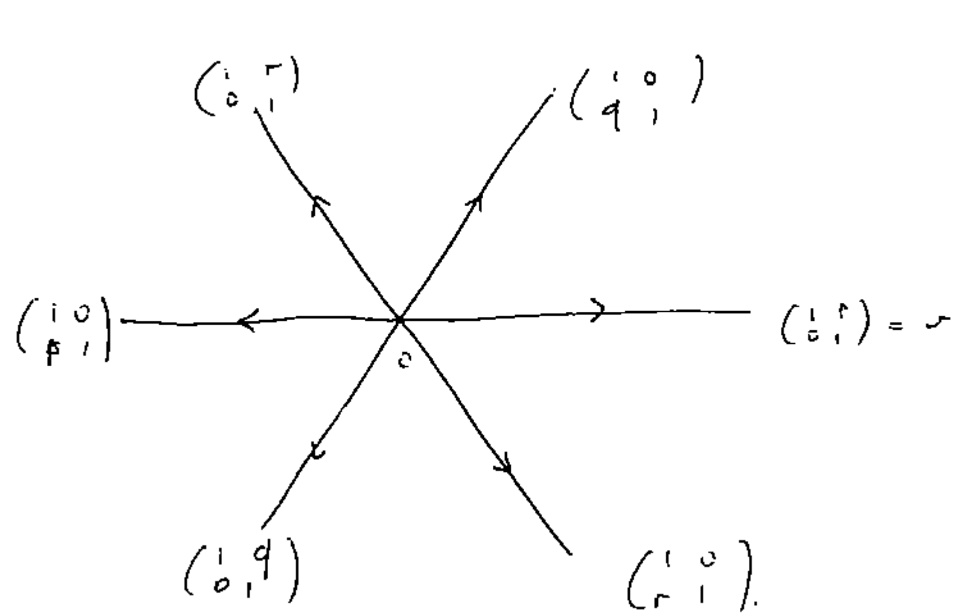
Step 4 Painleve Theory.

Where does PII come into the picture?

Now there is a wonderful way (going back essentially to Ablowitz & Segur, then Flaschka - Newell and Jimbo Ueno) to solve PII

$$u'' = 2u^3 + xu$$

Let p, q, r be 3 #'s satisfying $p+q+r+pqr = 0$
Consider Σ consisting of 6 rays with v attached as follows:



Let Ψ solves the following RHP:

- Ψ analytic in $\mathbb{C} \setminus \Sigma$
- $\Psi_+(z) = \Psi_-(z)\, v$ on Σ =
- $m = \Psi \begin{pmatrix} e^{i\left(\frac{4z^3}{3} + xz\right)} & 0 \\ 0 & e^{-i\left(\frac{4z^3}{3} + xz\right)} \end{pmatrix} \to I$ as $z \to \infty$

Then if we put

$$m = I + \frac{m_1(x)}{z} + O\left(\frac{1}{z^2}\right),$$

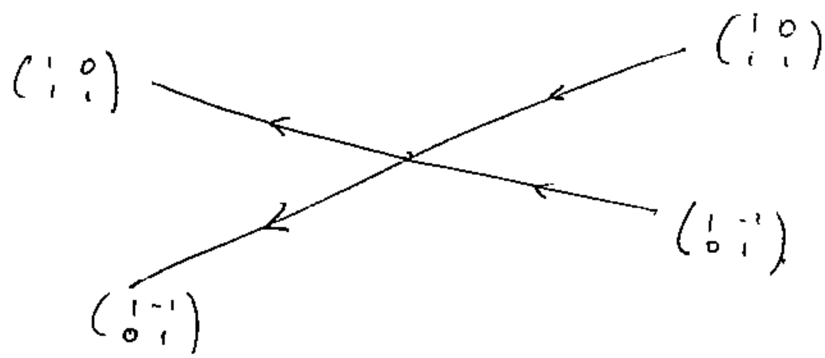
$$u(x) \equiv 2i\, (m_1(x))_{12}$$

solves PII.

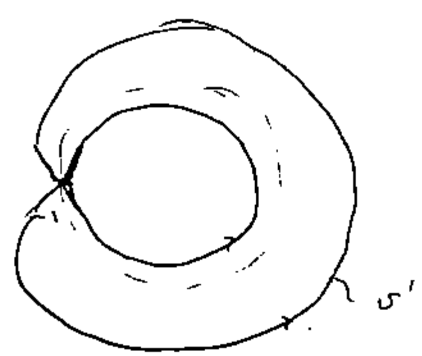
Of particular interest is the case when

$$p = 1 = -q \quad , \; r = 0$$

which can be deformed to a problem of the form.



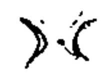
Now it turns out, at least for $\frac{2\sqrt{\lambda}}{k+1} \leq 1$, the RHP for Y is equivalent to a RHP on a contour of the following form



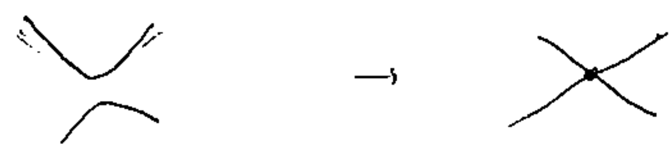
for $\frac{2\sqrt{\lambda}}{k+1} \sim 1$

Answer b. ... Dete ... the

leading contribution then comes for $z = -1$



Rotating by $90°$ we get



which is ~~the contour~~ precisely the RHP, with the right jump matrices for PII!

In this way PII comes into the picture.

Analytical remark: what kind of an analytical problem is this really.

Inhomogeneous problem $m = (1 - C_{w_{x,t}})^{-1} I$