An outline for deriving the formula for $d\lambda/d\varepsilon$ on p520

Recall that A is a square matrix that has a simple eigenvalue λ_1. For small ε let $\lambda(\varepsilon)$ be the smoothly varying eigenvalue of $A + \varepsilon B$ that equals λ_1 when $\varepsilon = 0$.

(a) First consider the case when A is block diagonal, say

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \tilde{A} \end{bmatrix}$$

where $\tilde{A}$ has dimension one less than A; show that

$$\frac{d\lambda}{d\varepsilon}(0) = b_{11},$$

the 1,1-entry of the perturbing matrix B.

Hint: $\lambda(\varepsilon)$ solves the implicit equation

$$f(\lambda, \varepsilon) = \det(A + \varepsilon B - \lambda I) = 0.$$

Now

$$\frac{d\lambda}{d\varepsilon}(0) = -\frac{\partial f/\partial\varepsilon(\lambda_1, 0)}{\partial f/\partial\lambda(\lambda_1, 0)}.$$

Recalling the proof of Proposition C.3.1, show that

$$\partial f/\partial\lambda(\lambda_1, 0) = -\det(\tilde{A} - \lambda_1 I) \neq 0,$$

and argue that

$$\partial f/\partial\varepsilon(\lambda_1, 0) = b_{11}\det(\tilde{A} - \lambda_1 I),$$

all other contributions to $\det(A + \varepsilon B - \lambda_1 I)$ being $\mathcal{O}(\varepsilon^2)$.

(b) For general A, let S be a matrix whose first column equals the eigenvector of A with eigenvalue λ_1 and whose remaining columns span range $(A - \lambda_1 I)$, the sum of all the eigenspaces with eigenvalues different from λ_1. Show that $S^{-1}AS$ has the above block-diagonal structure. *(Why is S invertible?)*

(c) By construction the first column of S, say $\mathbf{v}$, is an eigenvector of A with eigenvalue λ_1. Argue that $\mathbf{w}$, the first row of S^{-1}, is orthogonal to the range of $A - \lambda_1 I$ and satisfies $\langle \mathbf{w}, \mathbf{v} \rangle = 1$. Show that the 1,1-entry of $S^{-1}(A + \varepsilon B)S$ equals $\langle \mathbf{w}, (A + \varepsilon B)\mathbf{v} \rangle$ to conclude

$$d\lambda/d\varepsilon(0) = \langle \mathbf{w}, B\mathbf{v} \rangle.$$