7.11 Appendix: Index theory in two dimensions

The *index* of a closed curve Γ relative to a $\mathcal{C}^1$ vector field $\mathbf{F} : \mathcal{U} \to \mathbb{R}^2$ is a useful construct for the purpose of analyzing global behavior of planar systems. In this section, f and g will denote the components of $\mathbf{F}$, and we shall analyze systems of the form

$$x' = f(x, y) \qquad \text{and} \qquad y' = g(x, y). \tag{7.110}$$

Recall from Section 7.1 that a *Jordan curve* is a simple, closed[35] curve. Much of the theory in this section relies upon a famous result regarding Jordan curves in the plane:

Theorem 7.11.1. *(Jordan Curve Theorem): Every Jordan curve Γ in the plane separates $\mathbb{R}^2$ into two disjoint, open, connected sets, both of which have Γ as their boundary. One region is bounded and simply connected, while the other is neither bounded nor simply connected.*

The Jordan Curve Theorem seems rather intuitive for the closed curves that spring immediately to one's mind, like the circle or $\mathcal{C}^1$ deformations of it. However, Jordan curves can be incredibly intricate creatures, making it rather difficult to prove Theorem 7.11.1 rigorously. For sketches of complicated Jordan curves and a "modern" proof of the Jordan Curve Theorem, we refer you to the article, *"The Jordan Curve Theorem, Formally and Informally"*, by T. C. Hales, American Mathematical Monthly, Volume 114 (2007), pp. 882–894.

7.11.1 The index of a curve

Suppose that $\mathbf{F}$ is defined in some neighborhood of a Jordan curve Γ, and that there are no zeros of $\mathbf{F}$ on Γ. The *index* of Γ relative to $\mathbf{F}$ is an integer $I_{\mathbf{F}}(\Gamma)$ that measures the winding of the vector field $\mathbf{F}$ as Γ is traversed exactly once in the counterclockwise direction. More explicitly (see Figure 7.22), for $(x, y) \in \Gamma$, let

$$\Theta(x, y) = \arctan\left(\frac{g(x, y)}{f(x, y)}\right) \tag{7.111}$$

be the angle formed by the vector $\mathbf{F}(x, y)$ and the positive x-axis. Since $\tan(\phi + k\pi) = \tan(\phi)$ for any real ϕ and for any integer k, the inverse tangent function is multi-valued. Often, arctan is made single-valued by defining a principal value, but this is *not* the usage intended in (7.111). Rather, we choose a point $(x_0, y_0) \in \Gamma$ and require

[35] Phrased more technically, a Jordan curve in the plane is the image of a one-to-one, continuous map of the unit circle into the plane.

that arctan is continuous as (x,y) varies over $\Gamma \sim \{(x_0,y_0)\}$. For example, suppose

$$\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{7.112}$$

and Γ is the unit circle $\{x^2+y^2=1\}$, and let $(x_0,y_0)=(1/\sqrt{2},1/\sqrt{2})$. We may parametrize Γ by the function

$$\gamma(t) = (\cos(t+\pi/4), \sin(t+\pi/4)), \qquad 0 \le t < 2\pi.$$

If we define $\Theta(1/\sqrt{2},1/\sqrt{2})$ as a principal value, then at any point $\gamma(t) \in \Gamma$,

$$\Theta(\gamma(t)) = t + \pi/4.$$

Note that y/x is undefined at the two points $(0,\pm 1)$ on Γ—i.e., at $t=\pi/4$ and $5\pi/4$, but $\arctan(y/x)$ is nonsingular there.

Since $\lim_{t\to 2\pi^-} \Theta(t) \neq \Theta(0)$, the function Θ is not continuous on Γ. In words, the discontinuity $\Delta\Theta = \Theta(2\pi^-) - \Theta(0^+)$ equals the net change in the argument of $\mathbf{F}(x,y)$ over one counterclockwise cycle of Γ. The index of Γ relative to $\mathbf{F}$ is defined in terms of this discontinuity:

$$I_{\mathbf{F}}(\Gamma) = \frac{\Delta\Theta}{2\pi}. \tag{7.113}$$

Thus, the index of the unit circle with respect to the vector field $\mathbf{F}(x,y)=(x,y)$ equals $+1$.

More generally, let $\Delta\Theta$ denote the net change in Θ, defined by (7.111), over one counterclockwise cycle of a Jordan curve Γ. The index $I_{\mathbf{F}}(\Gamma)$ is defined by (7.113). Figure 7.22 illustrates this concept, by showing vectors $\mathbf{F}(x,y)$ (normalized for convenience) at 12 different chronologically-labeled points during a single counterclockwise trip around Γ. Relocating the numbered vectors so that they are all anchored at the origin (right panel of figure) makes it easier to observe that the vectors complete two clockwise cycles as we follow them in increasing order. Thus, in this case, $\Delta\Theta = -4\pi$, and $I_{\mathbf{F}}(\Gamma) = \Delta\Theta/(2\pi) = -2$.

Issues of multiple-valued functions may be avoided by calculating the index of Γ analytically as follows:

Lemma 7.11.2. *Let Γ be a $\mathcal{C}^1$ Jordan curve contained in an open set $\mathcal{U}$, and let $\mathbf{F}:\mathcal{U}\to\mathbb{R}^2$ be a $\mathcal{C}^1$ vector field. Then the index of Γ relative to the vector field $\mathbf{F}$ is given by*

$$I_{\mathbf{F}}(\Gamma) = \frac{\Delta\Theta}{2\pi} = \frac{1}{2\pi}\oint_{\Gamma} \frac{f\,dg - g\,df}{f^2+g^2}. \tag{7.114}$$

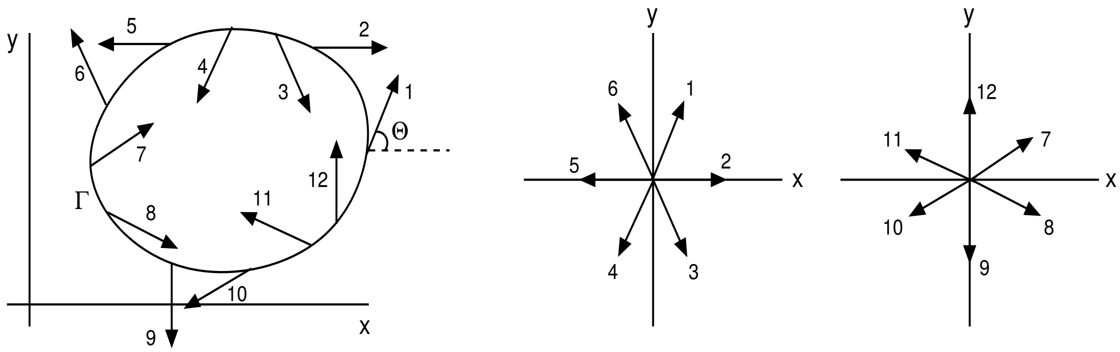

Figure 7.22: *A Jordan curve* Γ *in a smooth vector field* $\mathbf{F}(x, y)$. *The angle* Θ *formed by* $\mathbf{F}(x, y)$ *relative to the positive* x*-axis varies continuously as* Γ *is traversed. Following the vectors* 1 *through* 12 *in increasing order, these vectors complete two clockwise cycles during one counterclockwise cycle along* Γ. *Hence,* $\Delta\Theta = -4\pi$ *and the curve* Γ *has index* -2.

Proof. Using (7.111),

$$\Delta\Theta \;=\; \oint_\Gamma d\Theta \;=\; \oint_\Gamma d\arctan\left(\frac{g(x,y)}{f(x,y)}\right) \;=\; \oint_\Gamma \frac{f dg - g df}{f^2 + g^2}.$$

□

Remark: This Lemma is easily extended to the case in which Γ is *piecewise* $\mathcal{C}^1$.

It is illuminating to test out Lemma 7.11.2 on vector fields $\mathbf{F}$ associated with the linear systems covered in Chapter 2. For instance, let's use the lemma to recompute the index of the unit circle relative to the vector field (7.112) considered above. Let Γ be the unit circle parametrized by $\gamma(t) = (\cos 2\pi t, \sin 2\pi t)$, a Jordan curve which is traversed once in the counterclockwise direction as t increases from 0 to 1. By Lemma 7.11.2, the index of Γ relative to this vector field is

$$\frac{1}{2\pi}\oint_\Gamma \frac{x dy - y dx}{x^2 + y^2} \;=\; \frac{1}{2\pi}\int_0^1 \frac{2\pi\cos^2(2\pi t) + 2\pi\sin^2(2\pi t)}{\cos^2(2\pi t) + \sin^2(2\pi t)}\, dt \;=\; 1,$$

in agreement with our earlier calculation.

Reversing the direction of all vectors in a field $\mathbf{F}$ does not affect the index of Γ. If Γ denotes the unit circle as in the above example, then

- If $\mathbf{F}(x, y) = (-x, -y)$, then $I_{\mathbf{F}}(\Gamma) = 1$. The origin is a global attractor for the system (7.110).
- If $\mathbf{F}(x, y) = (-y, x)$, then $I_{\mathbf{F}}(\Gamma) = 1$ once again. The origin is stable but not attracting, and Γ happens to correspond to a periodic orbit.

- If $\mathbf{F}(x,y) = (-x,y)$, then $I_{\mathbf{F}}(\Gamma) = -1$. The origin is a saddle for the system (7.110).
- If
$$\mathbf{F}(x,y) = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x - \beta y \\ \beta x + \alpha y \end{bmatrix},$$
with $\beta \neq 0$, then the origin is either a stable focus (if $\alpha < 0$), center (if $\alpha = 0$), or an unstable focus (if $\alpha > 0$) for the system (7.110). In any case, $I_{\mathbf{F}}(\Gamma) = 1$.

The values of $I_{\mathbf{F}}(\Gamma)$ computed in the above example would have been the same regardless of the radius of the circle Γ. In fact, the index would remain unchanged even if Γ were continuously deformed into any other Jordan curve enclosing the origin. These statements are made more precise by the following series of propositions which, in turn, offer considerable information regarding global behavior of (7.110).

Proposition 7.11.3. *Let* $\mathbf{F}$ *be a* $\mathcal{C}^1$ *vector field in the plane*[36] *and let* Γ *be a piecewise smooth Jordan curve. If there are no zeros of* $\mathbf{F}$ *on* Γ *or in its interior, then* $I_{\mathbf{F}}(\Gamma) = 0$.

Proof. By Lemma 7.11.2 (see also the Remark that follows that Lemma),

$$I_{\mathbf{F}}(\Gamma) = \frac{1}{2\pi} \oint_{\Gamma} \frac{f\,dg - g\,df}{f^2+g^2} = \frac{1}{2\pi} \oint_{\Gamma} \frac{f g_x - g f_x}{f^2+g^2}\,dx + \frac{f g_y - g f_y}{f^2+g^2}\,dy.$$

By our hypotheses, $f^2 + g^2 \neq 0$ on or inside Γ, so we may apply Green's Theorem to obtain

$$I_{\mathbf{F}}(\Gamma) = \frac{1}{2\pi} \iint_{\mathrm{Int}(\Gamma)} \frac{\partial}{\partial x}\left(\frac{f g_y - g f_y}{f^2+g^2}\right) - \frac{\partial}{\partial y}\left(\frac{f g_x - g f_x}{f^2+g^2}\right) dA. \qquad (7.115)$$

The integrand actually reduces to 0 after the tedious process of computing the partial derivatives. □

Proposition 7.11.4. *Suppose* $\mathbf{F}$ *is a* $\mathcal{C}^1$ *vector field in the plane, and that* Γ_1 *and* Γ_2 *are Jordan curves. If* Γ_1 *can be continuously deformed to* Γ_2 *without passing through any zeros of* $\mathbf{F}$, *then* $I_{\mathbf{F}}(\Gamma_1) = I_{\mathbf{F}}(\Gamma_2)$.

Proof. Referring to Lemma 7.11.2, the index varies continuously as Γ_1 is continuously deformed to Γ_2. Since the index is integer-valued, the only way the index could vary continuously is if its value remains constant, implying that Γ_1 and Γ_2 have the same index. □

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[36]More generally, this Proposition holds if $\mathbf{F} : \mathcal{U} \to \mathbb{R}^2$ is $\mathcal{C}^1$ on an open, simply connected domain $\mathcal{U}$ that contains Γ.

Proposition 7.11.5. *If Γ happens to be the orbit of a periodic solution of $\mathbf{x}' = \mathbf{F}(\mathbf{x})$, then $I_{\mathbf{F}}(\Gamma) = 1$.*

The validity of the Proposition can be argued heuristically as follows: At any point $\mathbf{x}$ on Γ, the vector $\mathbf{F}(\mathbf{x})$ is tangent to the graph of Γ. Therefore, during one counterclockwise trip around Γ, the vector $\mathbf{F}(\mathbf{x})$ must spin once counterclockwise, so that $\Delta\Theta = 2\pi$ and $I_{\mathbf{F}}(\Gamma) = 1$. Here is a technical proof, originally given by H. Hopf[37].

Proof. Suppose that Γ as described in the hypothesis of the Proposition. By translating and rotating coordinates as needed, we may arrange that Γ is contained within the first quadrant of the xy plane and is tangent to the x-axis at some point $\mathbf{x}_0$ (see Figure 7.23a). Let $\mathbf{x}(t)$ be the solution of $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ with initial conditions $\mathbf{x}(0) = \mathbf{x}_0$. Re-parametrize $\mathbf{x}(t)$ by arc length s, using the counterclockwise orientation, P as the initial point (i.e., when $s = 0$), and scaling length so that $s = 1$ corresponds to one complete trip around the curve. Let

$$T = \{(s_1, s_2) : 0 \leq s_1 \leq s_2 \leq 1\}$$

denote the closed, triangular domain shown in Figure 7.23b, and define a vector field $\mathbf{G}$ on T as follows. First, if $(s_1, s_2) \neq (0, 1)$ and $0 \leq s_1 < s_2 \leq 1$, define

$$\mathbf{G}(s_1, s_2) = \frac{\mathbf{x}(s_2) - \mathbf{x}(s_1)}{|\mathbf{x}(s_2) - \mathbf{x}(s_1)|},$$

a normalized version of the vector depicted in Figure 7.23a. Next, along the diagonal $0 \leq s_1 = s_2 \leq 1$, define

$$\mathbf{G}(s_1, s_1) = \frac{\mathbf{F}(\mathbf{x}(s_1))}{|\mathbf{F}(\mathbf{x}(s_1))|},$$

the unit tangent vector to the curve at the point $\mathbf{x}(s_1)$. Finally, at the point $(s_1, s_2) = (0, 1)$, define

$$\mathbf{G}(0, 1) = -\frac{\mathbf{F}(\mathbf{x}_0)}{|\mathbf{F}(\mathbf{x}_0)|}.$$

As an exercise, we ask you to verify that $\mathbf{G}$ is continuous and non-zero on T. By Proposition 7.11.3, ∂T forms a curve whose index relative to $\mathbf{G}$ is zero. Now examine the angular variation of $\mathbf{G}$ along each of the three edges of ∂T during one counterclockwise circuit. Let $\Theta(s_1, s_2)$ denote the angle of $\mathbf{G}(s_1, s_2)$ relative to the positive x-axis, and observe that $\Theta(0, 0) = 0$ by our choice of $\mathbf{x}_0$. Assuming that Γ is oriented counterclockwise, then

[37]Not to be confused with E. Hopf, who is associated with the bifurcation phenomena discussed in Sections 8.7-8.8 of our text.

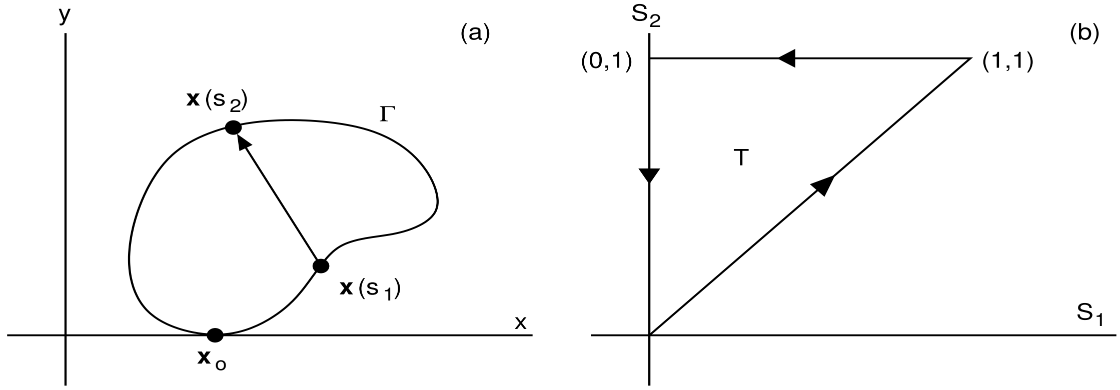

Figure 7.23: *Users guide to the proof of Proposition 7.11.5. (a) Parametrizing a periodic orbit* Γ *by arc length* s *and illustrating the definition of the vector field* $\mathbf{G}$ *in the proof. (b) The triangular domain* T *over which the vector field* $\mathbf{G}$ *is defined.*

- Along the diagonal edge, $\Theta(1,1) - \Theta(0,0) = 2\pi I_{\mathbf{F}}(\Gamma)$, recalling our earlier remark about $\mathbf{G}$ representing the unit tangent vector along that edge.
- Continuing right-to-left along the top edge, $\Theta(0,1) - \Theta(1,1) = -\pi$, by construction.
- Finally, moving top-to-bottom along the left edge, $\Theta(0,0) - \Theta(0,1) = -\pi$ as well.

Thus, $2\pi I_{\mathbf{F}}(\Gamma) - \pi - \pi = 0$, from which the Proposition follows. A similar argument holds if Γ is oriented clockwise. □

Here is an immediate consequence of Proposition 7.11.5, a stronger version of which appears later in this section:

Corollary 7.11.6. *If* Γ *is the orbit of a periodic solution of* $\mathbf{x}' = \mathbf{F}(\mathbf{x})$*, then* Γ *encloses at least one equilibrium point in its interior.*

Proof. Combine Propositions 7.11.3 and 7.11.5. □

7.11.2 The index of an equilibrium

If $\mathbf{x}^*$ is an isolated equilibrium of $\mathbf{x}' = \mathbf{F}(\mathbf{x})$, the *index* $I_{\mathbf{F}}(\mathbf{x}^*)$ is defined as $I_{\mathbf{F}}(\Gamma)$, where Γ is any Jordan curve such that (i) $\mathbf{x}^*$ is interior to Γ and (ii) there are no other equilibria on or inside Γ. Recall our above example with the vector field $\mathbf{F}(x,y) = (x,y)$, whose only zero is at the origin. If Γ is any circle centered at the origin, then $I_{\mathbf{F}}(\Gamma) = 1$, and therefore we may write $I_{\mathbf{F}}(\mathbf{0}) = 1$.

If an isolated equilibrium $\mathbf{x}^*$ is *non-degenerate* in the sense that 0 is not an eigenvalue of $\mathbf{DF}(\mathbf{x}^*)$, then the index of $\mathbf{x}^*$ is not affected by linearizing the system. Assuming without loss of generality that $\mathbf{x}^* = \mathbf{0}$, let us write $\mathbf{F}(\mathbf{x}) = A\mathbf{x} + \mathbf{r}(\mathbf{x})$, where $A = \mathbf{DF}(\mathbf{0})$ has non-zero determinant and $\mathbf{r}(\mathbf{x}) = o(|\mathbf{x}|)$ as $|\mathbf{x}| \to 0$ (see also the proof of Theorem 6.1.1). Let $\mathbf{L}(\mathbf{x}) = A\mathbf{x}$ be the vector field defined by the linearization of $\mathbf{F}$.

Proposition 7.11.7. *Under the above hypotheses,* $I_{\mathbf{F}}(\mathbf{0}) = I_{\mathbf{L}}(\mathbf{0})$.

Proof. It suffices to prove that if $\delta > 0$ is sufficiently small, then the [nonzero] vectors $\mathbf{F}(\mathbf{x})$ and $\mathbf{L}(\mathbf{x})$ never have opposite direction at any point $\mathbf{x}$ on the circle $|\mathbf{x}| = \delta$. (After all, this would assure that the vectors $\mathbf{F}$ and $\mathbf{L}$ could not "wind around" a different number of times during one counterclockwise trip around this circle.) Our proof is indirect: Suppose that $\mathbf{F}(\mathbf{x})$ and $\mathbf{L}(\mathbf{x})$ have opposite direction at some point $\mathbf{x}_0$ on the circle $|\mathbf{x}| = \delta$. Then there exists a constant $c > 0$ such that $\mathbf{F}(\mathbf{x}_0) = -c\mathbf{L}(\mathbf{x}_0)$, and therefore

$$|\mathbf{r}(\mathbf{x}_0)|^2 \;=\; |\mathbf{F}(\mathbf{x}_0) - \mathbf{L}(\mathbf{x}_0)|^2 \;=\; |-c\mathbf{L}(\mathbf{x}_0) - \mathbf{L}(\mathbf{x}_0)|^2 \;=\; (1+c)^2|\mathbf{L}(\mathbf{x}_0)|^2.$$

Because $\det(A) \neq 0$, the linear mapping $\mathbf{L}(\mathbf{x}) = A\mathbf{x}$ has only one zero, namely $\mathbf{x} = \mathbf{0}$. The mapping is also continuous, implying that

$$m = \min_{|\mathbf{x}|=1} |\mathbf{L}(\mathbf{x})|$$

exists and is positive. Linearity of the mapping then guarantees that the bound $|\mathbf{L}(\mathbf{x})| \geq m\delta$ holds for each $\mathbf{x}$ on the circle $|\mathbf{x}| = \delta$.

Combining this lower bound on $|\mathbf{L}(\mathbf{x})|$ with our earlier expression for $|\mathbf{r}(\mathbf{x}_0)|$,

$$|\mathbf{r}(\mathbf{x}_0)|^2 \;\geq\; (1+c)^2 m^2\delta^2 > m^2\delta^2,$$

and therefore

$$\frac{|\mathbf{r}(\mathbf{x}_0)|}{\delta} \geq m > 0.$$

This contradicts our assumption that $|\mathbf{r}(\mathbf{x}_0)| = o(\delta)$, because m does not depend on δ. □

The next Proposition states that the index of a non-degenerate equilibrium is -1 if the equilibrium is saddle or $+1$ otherwise.

Proposition 7.11.8. *Under the same hypotheses that precede Proposition 7.11.7,* $I_{\mathbf{F}}(\mathbf{0}) = \operatorname{sign}(\det A)$.

Proof. By Proposition 7.11.7, it suffices to compute the index of the origin for the linearized vector field $L(\mathbf{x}) = A\mathbf{x}$; we will write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for later convenience. To this end, $I_{\mathbf{L}}(\mathbf{0}) = I_{\mathbf{L}}(\Gamma)$, where Γ is the unit circle parametrized by $x(t) = \cos t$, $y(t) = \sin t$, $0 \leq t \leq 2\pi$. By Lemma 7.11.2,

$$\begin{aligned} I_{\mathbf{L}}(\Gamma) &= \frac{1}{2\pi} \oint_{\Gamma} d \arctan \left(\frac{cx(t) + dy(t)}{ax(t) + by(t)} \right) \\ &= \frac{\det A}{2\pi} \int_0^{2\pi} \frac{1}{(a\cos t + b\sin t)^2 + (c\cos t + d\sin t)^2}\, dt. \end{aligned} \tag{7.116}$$

Rarely is it productive to attempt direct evaluation of such integrals and, for that reason, we'll take a different tack. The RHS of (7.116) varies continuously with respect to a, b, c, d so long as $\det A = ad - bc \neq 0$. Moreover, since the index is *integer*-valued, continuity requires that its value remain *constant* if a, b, c, d are varied while preserving $\det A \neq 0$. If $\det A > 0$ and $ad > 0$, then letting $a \to d$ and $b, c \to 0$ preserves the sign of the determinant and facilitates computation of (7.116)—the result is $I_{\mathbf{L}}(\Gamma) = +1$. If $\det A > 0$ and $ad \leq 0$ (which forces $bc < 0$), the sign of the determinant is preserved if we increase ad until it becomes positive, after which the previous case can be recycled to establish that $I_{\mathbf{L}}(\Gamma) = +1$. The case $\det A < 0$ is handled similarly and is left as an exercise. □

We remark that an isolated equilibrium still has an index, even if the equilibrium is degenerate. Suppose that $\mathbf{F}$ is a smooth vector field with an isolated, non-degenerate zero at the origin, and let $\mathbf{G}$ be the vector field obtained by multiplying both components of $\mathbf{F}$ by $(x^2 + y^2)$. Then it is easy to check that the origin is an isolated, degenerate zero of $\mathbf{G}$, and $I_{\mathbf{F}}(\mathbf{0}) = I_{\mathbf{G}}(\mathbf{0})$.

As an application of Proposition 7.11.8, recall the scaled Lotka-Volterra equations (1.39). There are two [non-degenerate] equilibria: the origin is a saddle and, less transparently, $(1, 1)$ is a center (see Exercise 3 in Chapter 1 of our book). By the proposition, the origin has index -1 and $(1, 1)$ has index $+1$. If you're scratching your head as to why such information might be beneficial, fear not—an important theorem is coming at the beginning of the next subsection. One last stepping stone remains: describing the index of a curve that encloses multiple isolated equilibria.

Proposition 7.11.9. *Suppose* $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n$ *are isolated equilibria associated with a* C^1 *vector field* $\mathbf{F}$ *in the plane. If* Γ *is a Jordan curve containing these equilibria in* *its interior, then*

$$I_{\mathbf{F}}(\Gamma) = \sum_{i=1}^{n} I_{\mathbf{F}}(\mathbf{x}_i).$$

Proof. We sketch the proof for the special case of $n = 2$ equilibria, from which the general case follows immediately. Because the two equilibria are isolated and contained on the interior of Γ, it is possible to construct two disjoint circles centered at the equilibria and contained inside Γ (see Figure 7.24). Cut the Jordan curve Γ into two piecewise smooth Jordan curves along the dashed lines and circles, resulting in two piecewise smooth Jordan curves as illustrated in the figure. Let $J_{upper} = \Gamma_u \cup A_u \cup \cdots \cup E_u$, denote the Jordan curve in the "upper half" of the figure, and let J_{lower} (defined analogously) denote the Jordan curve in the lower half of the figure. By Proposition 7.11.3, both J_{upper} and J_{lower} have index zero because they enclose no equilibria. The indices of J_{upper} and J_{lower} are also equal to the sum of the changes in the angle Θ over each of the smooth arcs whose unions form those curves:

$$\begin{aligned}
\Delta\Theta(J_{upper}) &= \Delta\Theta(\Gamma_u) + \Delta\Theta(A_u) + \Delta\Theta(B_u) + \Delta\Theta(C_u) + \Delta\Theta(D_u) + \Delta\Theta(E_u) \;=\; 0 \\
\Delta\Theta(J_{lower}) &= \Delta\Theta(\Gamma_l) + \Delta\Theta(A_l) + \Delta\Theta(B_l) + \Delta\Theta(C_l) + \Delta\Theta(D_l) + \Delta\Theta(E_l) \;=\; 0.
\end{aligned}$$

Now convince yourself of each of the following:

- $\Delta\Theta(\Gamma) = \Delta\Theta(J_{upper}) + \Delta\Theta(J_{lower})$;
- $\Delta\Theta(A_u) = -\Delta\Theta(A_l)$ and similarly for the pairs C_u, C_l and E_u, E_l;
- Combining the preceding facts, the change in Θ during one trip around Γ must equal the negative of the change in Θ around the circular arcs formed by B_u, B_l, D_u, and D_l. The circles formed by B_u, B_l and by D_u, D_l are oriented *clockwise* and, as we complete one clockwise trip around each circle, $\Delta\Theta/2\pi$ measures the negative of the index of the equilibrium enclosed by the circle.
- Piecing everything together, the index of Γ must be the sum of the indices of the two equilibria.

□

7.11.3 Main Result

It's finally time to weave the above formalism into a rather powerful theorem that helps characterize two-dimensional flows, a generalization of Corollary 7.11.6:

Theorem 7.11.10. *Consider a planar system* $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ *such that (i)* $\mathbf{F} : \mathcal{U} \to \mathbb{R}^2$ *is* $\mathcal{C}^1$ *on the open, simply connected set* $\mathcal{U}$ *and (ii) any equilibria are isolated. If* $\gamma(t)$ *is a periodic solution of the system, then the interior of its orbit* Γ *must contain equilibria whose indices sum to* 1.

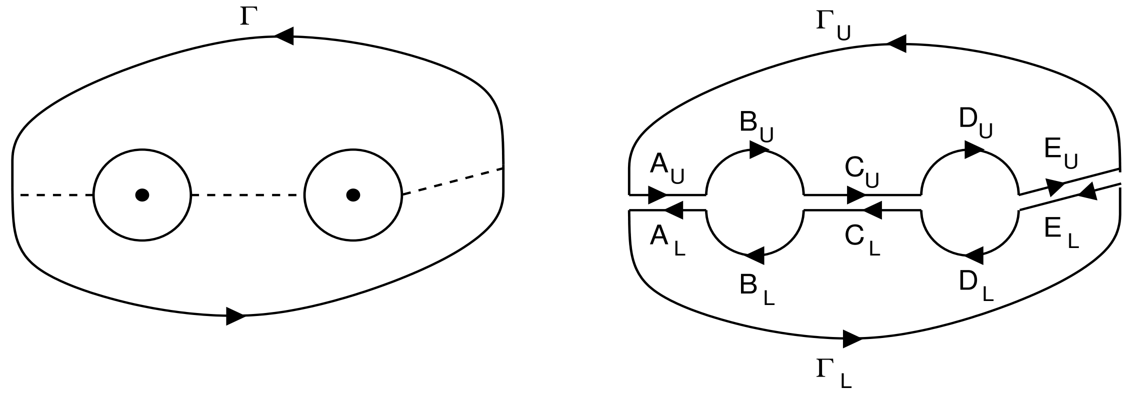

Figure 7.24: *Illustrating the proof of Proposition 7.11.9. Cut the Jordan curve* Γ *along the dashed lines to get the (U)pper and (L)ower pieces.*

Proof. Combine Propositions 7.11.5 and 7.11.9. □

Theorem 7.11.10 has a host of consequences. A periodic orbit Γ must enclose at least one equilibrium and, if the interior of Γ contains exactly one equilibrium, it cannot be a saddle (index -1). A periodic orbit cannot enclose an even number of hyperbolic equilibria.

Let us apply Theorem 7.11.10 to prove non-existence of periodic orbits in the system

$$x' = \alpha x - \gamma xy, \qquad y' = \beta y - \gamma xy, \tag{7.117}$$

where α, β, and γ are positive parameters. The system (7.117) can be interpreted as a crude model for population of two species in competition for the same food source. If species x is absent, then y grows exponentially with growth constant β, and if $y = 0$ then x grows exponentially with growth constant α. Both species are penalized equally by the term γxy, which is proportional to the product of the two populations. There are four nullclines in the phase plane: $x = 0$, $x = \beta/\gamma$, $y = 0$, and $y = \alpha/\gamma$, and two equilibria, the origin and $(x^*, y^*) = (\beta/\gamma, \alpha/\gamma)$. The Jacobian matrices associated with these equilibria are

$$\mathbf{DF}(0,0) \;=\; \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \qquad \mathbf{DF}(x^*, y^*) \;=\; \begin{bmatrix} 0 & -\beta \\ -\alpha & 0 \end{bmatrix}.$$

The origin is an unstable node since both eigenvalues of $\mathbf{DF}(0,0)$ are real and positive, and (x^*, y^*) is a saddle because $\det \mathbf{DF}(x^*, y^*) = -\alpha\beta < 0$. It follows that $I_{\mathbf{F}}(0,0) = 1$ and $I_{\mathbf{F}}(x^*, y^*) = -1$. If a periodic orbit (call it Γ) exists, Theorem 7.11.10 implies that Γ must enclose equilibria whose indices sum to 1, and the only way this is possible is if Γ encloses the origin but not (x^*, y^*). Certainly, such Γ would fail to be biologically relevant since it would include points with negative x

and y coordinates. In fact, we claim that there can be no periodic orbits even if we allow the possibility that $x < 0$ or $y < 0$. Any trajectory Γ enclosing the origin would cross both coordinate axes, violating the existence and uniqueness theorem because the axes themselves form solution trajectories. It follows that (7.117) cannot have periodic solutions.