Stable and Unstable Manifolds II

Remark: Before proceeding, we recommend that you familiarize yourself with basic XPP syntax via the introductory Chapter 1 examples `ch1-riccati.ode` and `ch1-van-der-Pol.ode` and their accompanying documentation.

The plain text file `ch6-ai.ode` is an XPP script for numerical solution of the reduced activator-inhibitor model equations

$$\begin{aligned} x' &= \sigma x^2/(1+y) - x \\ y' &= \rho[x^2 - y], \end{aligned}$$

where ρ and σ are positive parameters (See Sections 5.6 and 6.3.1 of our textbook for details).

The default parameter values, initial conditions, and viewing window are all specified in the `ch6-ai.ode` file. For the purposes of the following exercises, the default viewing window and parameter values serve as a useful starting point.

Here is how to use this XPP script to plot the stable and unstable manifolds at one of the equilibria:

1. Load the file `ch6-ai.ode` into XPP and issue the commands **N**ullcline, **N**ew. With the default parameters $\sigma = 2.1$ and $\rho = 1.1$, the non-trivial x-nullcline $y = \sigma x - 1$ crosses the y-nullcline $y = x^2$ at two points (denoted $\mathbf{P}_{\pm}$ in the text).

2. XPP can calculate (numerically) the stabilities of these equilibria, as well as the stable and unstable manifolds of a saddle point. From the main menu, enter **S**ing pts and then choose **(M)**ouse. XPP waits for you to click your mouse cursor in the vicinity of an equilibrium that you wish to study. Try clicking near the equilibrium $\mathbf{P}_{-}$. XPP then prompts you with two questions: "Print Eigenvalues?" and "Draw Invariant Sets?". Answer "Yes" to both questions. If you do this correctly, you should see [portions of] the unstable and stable manifolds of $\mathbf{P}_{-}$ as they appear in Figure 6.12(a) of the textbook. Additionally, a small XPP window appears showing the coordinates of the equilibrium you selected, as well as some basic information about eigenvalues of the Jacobian at that equilibrium. XPP uses the abbreviations $c+$, $c-$, im, $r+$, and $r-$ to indicate how many eigenvalues are complex with positive real part, complex with negative real part, pure imaginary, real and positive, or real and negative, respectively. In this example, you should see that $c+ = c- = \text{im} = 0$ and $r+ = r- = 1$, indicating that $\mathbf{P}_{-}$ is a saddle.

3. From the main menu, enter **E**rase. Keeping $\sigma = 2.1$, change ρ to 0.9, repeat the steps listed above, and notice how the stable and unstable manifolds change orientation. If you do this correctly, you should see a reproduction of Figure 6.12(b) in the textbook.

4. You may wish to create a slider bar that allows you to vary σ, say from 1 to 3, to observe the affect on the nullclines. (The parameter ρ has no effect on the nullclines, but it does affect the trajectories.) There are no non-trivial equilibria if $\sigma < 2$.

5. Remember that from the main menu, you may issue the commands **D**ir.field/flow, **(F)**low and then set "Grid" to be 5 or 10 in order to see a phase portrait for a given parameter set.

6. For more XPP documentation, be sure to refer to Bard Ermentrout's XPP website at

   `http://www.math.pitt.edu/~bard/xpp/xpp.html`