Stable and Unstable Manifolds I

Remark: Before proceeding, we recommend that you familiarize yourself with basic XPP syntax via the introductory Chapter 1 examples `ch1-riccati.ode` and `ch1-van-der-Pol.ode` and their accompanying documentation.

The plain text file `ch6-duffing.ode` is an XPP script for numerical solution of Duffing's equation

$$\begin{aligned} x' &= y \\ y' &= x - x^3 - \beta y \end{aligned}$$

where β is a non-negative friction coefficient (See Sections 1.4.2 and 6.6.4).

A default choice of β, initial conditions, and default viewing window are all specified in the `ch6-duffing.ode` file.

Here is how to use this XPP script to plot the stable and unstable manifolds at one of the equilibria:

1. Load the file `ch6-duffing.ode` into XPP and issue the commands **N**ullcline, **N**ew. There are three equilibria, located at the points where the cubic nullcline crosses the horizontal axis.

2. XPP can calculate (numerically) the stabilities of these equilibria, as well as the stable and unstable manifolds of a saddle point. From the main menu, enter **S**ing pts and then choose **(M)**ouse. XPP waits for you to click your mouse cursor in the vicinity of whichever equilibrium you wish to study. Try clicking near the equilibrium at the origin. XPP then prompts you with two questions: "Print Eigenvalues?" and "Draw Invariant Sets?". Answer "Yes" to both questions. If you do this correctly and use the default friction $\beta = 0.25$, you should see [portions of] the unstable and stable manifolds of the origin as they appear in Figure 6.10(b) of the textbook. Additionally, a small XPP window appears showing the coordinates of the equilibrium you selected, as well as some basic information about eigenvalues of the Jacobian at that equilibrium. XPP uses the abbreviations $c+$, $c-$, im, $r+$, and $r-$ to indicate how many eigenvalues are complex with positive real part, complex with negative real part, pure imaginary, real and positive, or real and negative, respectively. In this example, you should see that $c+ = c- = \text{im} = 0$ and $r+ = r- = 1$, indicating that the origin is a saddle. If you re-issue the commands **S**ing pts, **(M)**ouse and click near one of the other two equilibria, XPP will only ask one question: "Print eigenvalues?" Because the equilibria $(\pm 1, 0)$ are asymptotically stable, XPP will not ask whether you wish to draw invariant sets associated with those two equilibria.

3. From the main menu, enter **E**rase. Create a slider bar that allows you to vary β from 0 (no friction) to 0.25. Gradually reduce β to zero, and then mimic the steps above to plot the stable and unstable manifolds of the origin (which is still a saddle point). If you do this correctly, you should see a reproduction of Figure 6.10(a) in our textbook. The global stable and unstable manifolds of the origin coincide.

4. Remember that from the main menu, you may issue the commands **D**ir.field/flow, **(F)**low and then set "Grid" to be 5 in order to see a phase portrait for a given choice of β.

5. For more XPP documentation, be sure to refer to Bard Ermentrout's XPP website at

`http://www.math.pitt.edu/~bard/xpp/xpp.html`