Evaluating Singular and Nearly Singular Integrals Numerically

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This work is supported by NSF VIGRE grant number DMS-9983320.

1 Abstract

We look for a method of numerical integration, with high-order accuracy, for functions with singularities of the type found in the function $\log|x|$ or near-singularities of the type found in the function $\log(x^2+a^2)$, where a is a small number. We apply standard rules of numerical integration that hold for smooth functions but fail in our case, then develop a correction term for the singularity or near-singularity that allows us to evaluate these integrals to third-order accuracy.

2 Introduction

This paper will address two related questions. In each case, we want to develop a numerical method of evaluating an integral $I = \int f(x)K(x)dx$, where f is a smooth function and K is a more troublesome function that possesses a singularity or a "near-singularity," as we will define below, somewhere along the region of integration.

First, we develop a numerical approximation with third-order accuracy for the integral of a function of the form $f(x)\log|x|$, where $f(x)$ is a sufficiently smooth function. This product, then, is a function with a singularity that is roughly as bad as the singularity of $\log|x|$ at 0.

The difficulty to the integration is that we are trying to take the integral on an interval containing 0. For all practical purposes, it is sufficient to choose $[-1,1]$ to be this interval, since away from the singularity the function is smooth, and there already exist good numerical rules for integrating such functions.

One such numerical rule is the Euler-Maclaurin summation formula, which we will refer to on multiple occasions in this paper. We state this formula as given by [3]: If $g(x)$ is a C^{2k+1} function, then

$$\int_1^N g(x)dx = \sum_{i=1}^{N-1} g(i) + \tfrac{1}{2}[g(N)-g(1)] + \sum_{j=0}^{k} \frac{b_{2j}}{(2j)!}[g^{(2j-1)}(N) - g^{(2j-1)}(1)] + \int_1^N P_{2k+1}(x)g^{2k+1}(x)dx \tag{1}$$

where we write P_M for the periodic extension of the Mth Bernoulli polynomial B_M on $[0,1]$. (Recall that the Bernoulli numbers are given by $b_1 = -\frac{1}{2}, b_2 = \frac{1}{6}, b_4 = -\frac{1}{30}$, etc. and $b_3 = b_5 = b_7 = \ldots = 0$, and that the Bernoulli polynomials are defined using these so that $B_M(0) = b_M$.)

From Euler-Maclaurin, we can derive a quadrature rule that uses a weighted sum of the values of the function alone (not its derivatives) at different points that gives $O(h^l)$ accuracy for a C^l function. This is analogous to the trapezoid rule in that the weight on the interior of the interval is just 1 and the different weights come only at the endpoints; it differs from the trapezoid rule by giving higher-order accuracy for a sufficiently smooth function and the proper choice of weights. This rule will also be useful for our purposes, so we state it here:

Lemma 1 *Let $f(x)$ be a C^l function, let N be some large integer, and define a set of weights $w_j, -N \le j \le N$, using the Bernoulli numbers b_k by setting $w_{N-i} = w_{i-N} = 1$ for $l-1 \le i \le N$, and $w_{N-i} = w_{i-N} = 1 + a_i$ for $0 \le i \le l-2$, where a_i are the solutions to the system of equations,* *for integers s,* $0 < s < l-1$*:*

$$\sum_{i=0}^{i=l-2} i^s\, a_i \;=\; \frac{b_{s+1}}{s+1}, \qquad \sum_{i=0}^{i=l-2} a_i \;=\; b_1 \;=\; -\tfrac{1}{2} \tag{2}$$

Then if $h = 1/N$*, there is some constant* C *such that*

$$\left| \int_{-1}^{1} f(x)\, dx - \sum_{j=-N}^{N} w_j h\, f(jh) \right| \;\leq\; Ch^l \tag{3}$$

Effectively, this weighting system gives $l-1$ coefficients at each end of the interval $[-N, N]$ that are different from 1, with the rest of the interval having the same weight of 1. Thus, in the case where $l \;=\; 2$ we would have $w_N \;=\; w_{-N} \;=\; \frac{1}{2}$, otherwise $w_j \;=\; 1$ — simply a restatement of the trapezoid rule.

In the case $l \;=\; 3$ we get

$$w_N \;=\; w_{-N} \;=\; \frac{5}{12} \qquad w_{N-1} \;=\; w_{1-N} \;=\; \frac{13}{12} \qquad w_j \;=\; 1 \text{ for } |j| < N-1 \tag{4}$$

It is this third-order rule that we will use in the following discussions.

We will apply this rule naively to our singular integral to develop a finite sum approximation to it which will not be particularly good. Then we will find a correction term to it so that it provides an approximation of accuracy $O(h^3)$, where h is the step size in our finite sum approximation, for any function $f \in C^4$. We could extend our argument by introducing further corrections to give higher-order accuracy for sufficiently smooth functions f, but in practice the constants bounding the error term tend to become larger as we move to higher powers of h, and third-order accuracy will usually be sufficient for practical purposes, so we will content ourselves with this.

We then turn to the case where the troublesome part of the problem is not a true singularity but a near-singularity. In this case, we have a function $f(x)\, \log|x^2 + a^2|$, where $f(x)$ is smooth and a is very small – so small as to be significantly smaller than $1/N$ for a typical choice of N (once again writing N for the number of subintervals that we divide our intervals $[-1, 0]$ and $[0, 1]$ into). Thus, in theory we could apply the trapezoid rule or some form of the Euler-Maclaurin summation formula with sufficiently small step size to the situation, making N large compared to $1/a$, and we would have no need for any other rule. However, in practice, if a is very small, the number of steps required becomes very large, and so it is useful to have a rule that allows for step sizes significantly larger than a. Once again, we will develop a finite sum and a correction term as an approximation to functions of this form that gives accuracy of order $O(h^3)$, for any function $f \in C^4$.

Finally, we will include a series of tables that demonstrate the third-order accuracy of these approximations for some examples.

3 The Singular Case

We begin with the case of the singular integral: an integral of the form $I \;=\; \int_{-1}^{1} f(x)\, K(x) dx$, where $f(x)$ is a smooth function and $K(x)$ is a smooth function except at $x \;=\; 0$, where there is a singularity. In particular, we consider the case where $K(x) \;=\; \log|x|$, the natural logarithm function.

We want to find a finite sum that approximates an integral of this form. To this end, we use the set of weights defined in (2) to define the finite sum that will approximate our (singular) integral, at the same time introducing a correction term based on the value of the function $f(x)$ at the point of the singularity of $K(x)$, $x = 0$. Notice that the fact that $\log(0)$ is undefined requires us to omit the $j = 0$ term from our finite sum.

We intend to show the following:

Theorem 1 *Let $f : [-1, 1] \to \Re$ be a C^4 function, and define $I = \int_{-1}^{1} f(x) \log|x| \, dx$. Given any integer N, with $h = 1/N$, we define also the following sum as an approximation to the integral:*

$$S = \sum_{j=-N,\, j\neq 0}^{j=N} (f(jh) \log|jh| \, w_j \, h) + h \log\left(\frac{h}{2\pi}\right) f(0) \tag{5}$$

where w_j is defined as in (4). Then there is some constant C dependent on f but not on N such that $|S - I| \leq Ch^3$.

Proof: We will prove this theorem by first proving it in several special cases, which we state as lemmas, then use these as a foundation to establish the general result. We begin with the case where $f \equiv 1$, which we state via the following lemma.

Lemma 2 *Let $I = \int_{-1}^{1} \log|x| dx$. Given any integer N, with $h = 1/N$, we define*

$$S = \sum_{j=-N, j\neq 0}^{j=N} (\log|jh| \, w_j \, h) + h \log\left(\frac{h}{2\pi}\right) f(0) \tag{6}$$

Then there is some constant C such that $|S - I| \leq Ch^3$.

Proof: Because $\log|x|$ is an even function,

$$I = 2 \int_0^1 \log(x) dx = 2 \lim_{b\to 0} (x \log x - x)|_b^1 = -2 \tag{7}$$

To find S, we use a consequence of the Euler-Maclaurin summation formula applied to $f(x) = \log(x)$ derived by [6], p. 543, which we state here:

For any N, and for $M \geq 2$,

$$\log\left(\frac{N!}{\sqrt{2\pi N}\, N^N e^{-N}}\right) = \sum_{k=2}^{M} \frac{b_k}{k(k-1)N^{k-1}} \pm \int_N^\infty \frac{P_M(t)}{Mt^M} dt \tag{8}$$

where b_k is the kth Bernoulli number and P_M is the periodic extension of the Mth Bernoulli polynomial B_M defined on $[0, 1]$.

Now B_M is bounded on $[0, 1]$ by some constant C_1, and so the integral

$$\left|\int_N^\infty \frac{P_M(t)}{Mt^M} dt\right| \leq \frac{C_1}{M} \left|\int_N^\infty t^{-M} dt\right| \leq \left|\frac{C_1}{N^M}\right|$$

This gives us the following form of Stirling's formula:

$$\log(N!) \;=\; \tfrac{1}{2}\log(2\pi N) + N\log N - N + \sum_{k=2}^{M} \frac{b_k}{k(k-1)N^{k-1}} + O(N^{-M}) \tag{9}$$

Now we rewrite (6) to use this:

$$S \;=\; \sum_{j=-N, j\neq 0}^{N} (\log|jh|\, w_j\, h) + h\log\left(\frac{h}{2\pi}\right) f(0) \tag{10}$$

$$= \; 2\left(\sum_{j=1}^{N} (\log j + \log h)\, w_j\, h\right) + h\log\left(\frac{h}{2\pi}\right) f(0) \tag{11}$$

$$= \; 2\left(\sum_{j=1}^{N} h\log j + \sum_{j=1}^{N} h\log h + \sum_{i=0}^{l-1} a_i\, h\log((N-i)h)\right) + h\log\left(\frac{h}{2\pi}\right) f(0) \tag{12}$$

$$= \; 2\left(h\log(N!) + \log h + \sum_{i=1}^{i=l-1} a_i\, h\log(1-ih)\right) + h\log\left(\frac{h}{2\pi}\right) f(0) \tag{13}$$

where we have used the fact that $\log(Nh) = \log(1) = 0$. Now we use a Taylor expansion for $\log(1-ih)$ about $i=0$, where this function is C^∞, to give that for any l (we'll choose $l=3$),

$$\log(1-ih) \;=\; \sum_{j=1}^{l-1} -\frac{1}{j}(ih)^j + O(h^{l-1}) \tag{14}$$

Then substituting this into (13), we get

$$S \;=\; 2\left(h\log(N!) + \log h + \sum_{i=1}^{l-1}\sum_{j=1}^{l-1} -\frac{1}{j} a_i i^j h^{j+1} + O(h^l)\right) + h\log\left(\frac{h}{2\pi}\right) f(0) \tag{15}$$

$$S \;=\; 2\left(h\log(N!) + \log h + \sum_{j=1}^{l-1} \frac{-b_{j+1}}{j(j+1)} h^{j+1} + O(h^l)\right) + h\log\left(\frac{h}{2\pi}\right) f(0) \tag{16}$$

where we have used (2). Then substituting from (9) with $M = l-1 = 2$ gives

$$S \;=\; 2\left(\tfrac{1}{2}h\log(\frac{2\pi}{h}) - \log h - 1 + \sum_{k=2}^{l} \frac{b_k h^k}{k(k-1)} + O(h^l) + \log h - \sum_{j=1}^{l-1} \frac{b_{j+1}h^{j+1}}{j(j+1)} + O(h^l)\right)$$
$$+h\log\left(\frac{h}{2\pi}\right) \tag{17}$$

And this equation reduces easily to give (with (7))

$$S \;=\; -2 + O(h^l) \;=\; I + O(h^l) \tag{18}$$

This completes the proof of Lemma 1, establishing the veracity of Theorem 1 in the case where $f \equiv 1$.

We now prove Theorem 1 in another possible case, namely when f is odd. We state the desired result in the following lemma.

Lemma 3 *Suppose $f : [-1, 1] \to \Re$ is an odd function, and define I, S as in Theorem 1. Then $|S - I| = 0$.*

PROOF: Clearly $log|x|$ is even in x, and $w_j h \log |jh|$ is even in j, so $I = S = 0$.

Before we move on to the general case, a final lemma will be useful.

Lemma 4 *Suppose $f(x) = x^2$, and define I, S as in Theorem 1. Then there is some constant C such that $|S - I| \le Ch^3$.*

PROOF: We first find an exact value for I by integrating in general x^{2p} for any positive integer p by parts:

$$\int x^{2p} \log x \, dx \;=\; x^{2p-1}(x \log x - x) \, dx + \int x^{2p} \, dx + C \;=\; \frac{x^{2p}}{2p}(x \log x - x) - \int \frac{x^{2p}}{2p} \log x \, dx + \frac{x^{2p+1}}{2p+1} + C \tag{19}$$

Then bringing the middle term on the right side over to the left side and simplifying gives

$$\int x^{2p} \log x \, dx \;=\; \frac{x^{2p+1}}{2p+1} \log x - \frac{x^{2p+1}}{2p+1}^{\,2} + C \tag{20}$$

So in this particular case, where $p = 1$, we have

$$I = 2 \int_0^1 x^2 \log x \, dx = -\frac{2}{9} \tag{21}$$

Now we turn to the sum S. Note that $f(0) = 0$, so we may drop the correction term and consider only

$$S \;=\; 2 \sum_{j=1}^{N} \Big(w_j h (jh)^2 \log(jh) \Big) \;=\; 2h^3 \left(\sum_{j=1}^{N} j^2 \log j + \sum_{j=1}^{N} j^2 \log h - \frac{7}{12} \log(1) + \frac{1}{12} \log(1-h) \right) \tag{22}$$

Recall (for instance, see [6]) that for the Bernoulli polynomials B_M, we have

$$\sum_{j=1}^{N} j^{2p} \;=\; \frac{B_{2p+1}(N+1) - B_{2p+1}(0)}{2p+1} \;=\; \frac{B_{2p+1}(N+1)}{2p+1} \tag{23}$$

since $B_{2p+1}(0) = b_{2p+1} = 0$. This will give us an expression for the second sum in (22):

$$2h^3 \sum_{j=1}^{N} j^2 \log h = \left(\frac{2}{3} + h + \frac{h^2}{3} \right) \log h \tag{24}$$

Meanwhile, however, we turn to the first sum in (22), which we approximate by an integral. In this case, we are dealing with a function $g(x) = x^2 \log x$ on the interval [1,N], which does not contain the singularity at 0, and so $g(x) \in C^\infty$ on this interval, meaning that the Euler-Maclaurin summation formula (1) applies for any k. For our particular $g(x)$, we have:

$$g'(x) = 2x \log x + x \tag{25}$$

$$g''(x) = 2 \log x + 3 \tag{26}$$

$$g^{(3)}(x) = 2/x \tag{27}$$

$$g^{(4)}(x) = -2/x^2 \tag{28}$$

$$g^{(5)}(x) = 4/x^3 \tag{29}$$

So applying these in (1) with $k = 2$ gives

$$\int_1^N x^2 \log x \, dx = \sum_{j=1}^{N} j^2 \log j - \tfrac{1}{2}[g(N) - g(1)] + \frac{1}{12}[g'(N) - g'(1)] + \frac{1}{720}[g^{(3)}(N) - g^{(3)}(1)] + \int_1^N P_5(x-1) g^{(5)}(x) \, dx \tag{30}$$

Then substituting from (20) and noting that P_5 is bounded gives

$$\sum_{j=1}^{N} j^2 \log j = \left[\frac{x^3}{3} \log x - \frac{x^3}{9}\right]_1^N - 0 - \tfrac{1}{2} N^2 \log N + (2N \log N + N - 1)/12 + (2/N - 2)/720 + O(1)$$

$$= \frac{N^3}{3} \log N - \frac{N^3}{9} - \tfrac{1}{2} N^2 \log N + \frac{1}{6} N \log N + \frac{N}{12} + O(1) \tag{31}$$

This result then substitutes into (22) to give (with (24))

$$S = 2h^3 \left(\frac{N^3}{3} \log N - \frac{N^3}{9} - \tfrac{1}{2} N^2 \log N + \frac{1}{6} N \log N + \frac{N}{12} + O(1)\right) + \left(\frac{2}{3} + h + \frac{h^2}{3}\right) \log h + \frac{1}{6} h^3 \log(1-h) = -\frac{2}{9} + O(h^3) \tag{32}$$

So $|I - S| = O(h^3)$, as claimed, which concludes the proof of Lemma 4.

Equipped with this result, we can prove the theorem in the general case:

Proof of Theorem 1:Given any function $f \in C^4$, we can write a Taylor expansion for f of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + g(x) \tag{33}$$

where g is a C^4 function with the property that $g(0) = g'(0) = g''(0) = g^{(3)}(0) = 0$, and $a_0 = f(0)$.

Then we can write

$$I - S = a_0 \left(R_0(h) - h \log(\frac{h}{2\pi})\right) + a_1 R_1(h) + a_2 R_2(h) + a_3 R_3(h) + R_g(h) \tag{34}$$

where for $0 \le i \le 3$,

$$R_i(h) \;=\; \int_{-\infty}^{\infty} x^i \log|x|\, dx - \sum_{j=-\infty}^{\infty} \Big(h(jh)^i \log|jh| \Big) \tag{35}$$

and

$$R_g(h) \;=\; \int_{-\infty}^{\infty} g(x) \log|x|\, dx - \sum_{j=-\infty}^{\infty} \left(h\, g(jh) \log|jh| \right) \tag{36}$$

From Lemma 2, we know that there is some constant C_0 such that $|R_0(h) - h\log(\frac{h}{2\pi})| \le C_0 h^3$, and similarly from Lemma 4, we know that there is some constant C_2 such that $|R_2(h)| \le C_2 h^3$. Finally, Lemma 3 gives $R_1(h) = R_3(h) = 0$.

Moreover, we know that the first three derivatives of $g(x)\log|x|$ exist and are bounded because of the fact that $g(0) = g'(0) = g''(0) = g^{(3)}(0) = 0$, which does away with the singularity at the origin. So the Euler-Maclaurin summation formula (1) applies to give third-order accuracy: there is some constant C_g such that $|R_g(h)| \le C_g h^3$.

Thus, we have established that

$$\begin{aligned} |I - S| \;&\le\; \left| a_0 \left(R_0(h) - h\log(\frac{h}{2\pi}) \right) \right| + |a_1 R_1(h)| + |a_2 R_2(h)| \\ &\quad + |a_3 R_3(h)| + |R_g(h)| \\ &\le\; |a_0| C_0 h^3 + |a_2| C_2 h^3 + C_g h^3 \end{aligned} \tag{37}$$

And by setting $C = |a_0|C_0 + |a_2|C_2 + C_g$, the statement of the theorem follows.

4 The Nearly Singular Case

What we have done, then, is to establish a method of integrating past a singularity. But occasionally we will encounter functions (for instance, in the study of layer potentials) that, though technically continuous, nevertheless have large jumps or spikes that cause them to act as if they contained a singularity. That is, instead of an exact singularity, we have a "near-singularity" – a point where the value of the function is not infinite, but a very large finite number. In theory, our normal rules of integration for continuous functions should apply to functions of this kind, but in practice, the value may be so large at the point of near-singularity that any attempt to evaluate the integral by numerical methods with reasonable accuracy would require a prohibitively small step size – and, perhaps, more precision than computer roundoff error usually allows for. We will be considering integrals of the form $I = \int f(x)\log(x^2 + a^2)dx$; if a is a very small number, then it should be clear that the value of the integrand at the origin can be a large number.

We search, then, for a numerical rule similar to that we found in the previous section for singular integrals, which we state in the form of the following theorem.

Theorem 2 *Let $f : [-1,1] \to \Re$ be a C^4 function, let a be some small number, and define $I = \int_{-1}^{1} f(x)\log(x^2 + a^2)dx$. Given any integer N, such that $N \ll 1/a$, we define also the following sum as an approximation to the integral, with $h = 1/N$:*

$$S \;=\; \sum_{j=-N}^{N} \Big(f(jh)\log((jh)^2 + a^2) w_j\, h \Big) - 2h\log(1 - e^{-2\pi a/h}) f(0) \tag{38}$$

where w_j is defined as in (4). Then there is some constant C such that $|S - I| \le Ch^3$.

Proof: First, we state (without proof) a simple result from calculus, which we will use in the following discussion:

Lemma 5 *Let $f \in C^1(0,b)$, and suppose $\lim_{x\to 0^+} f'(x) = 0$. Then $\lim_{x\to 0^+} f(x)$ exists.*

We will also introduce the concept of a smooth cutoff function ζ. Essentially, we want to avoid dealing with the changes in the coefficients w_j at the endpoints of the interval $[-1,1]$, so we introduce some infinitely differentiable function ζ such that $\zeta \equiv 1$ on some interval $[-c,c]$ (where c is chosen independently of a) about 0, and $\zeta \equiv 0$ outside some other interval $[-b,b]$, where $c < b < 1$, so that all derivatives of ζ (or any multiple of ζ) near 1 or -1 are likewise 0. We also stipulate that ζ must be an even function.

Note also that the integral of ζ or any multiple thereof on $[1,\infty]$ or $[-1,-\infty]$ is zero, so if we are integrating a multiple of ζ, we can take the entire real line as our region of integration.

Then, if we multiply the integrand by this ζ function, we can make the following claim:

Lemma 6 *Given any positive number h and any infinitely differentiable test function ζ of the form above, we define the functions $I(h)$ and $S(h)$ by*

$$I(h) \;=\; \int_{-\infty}^{\infty} \zeta(x)\log(x^2+a^2)\,dx \tag{39}$$

$$S(h) \;=\; \sum_{j=-\infty}^{\infty} \Big(h\,\zeta(jh)\log((jh)^2+a^2)\Big) \tag{40}$$

Then there is some constant C, independent of both h and a, and some constant L, dependent only on the ratio a/h, such that $|S(h) - I(h) + Lh| \le Ch^3$.

Proof: First, we try to extract a factor of h from both the sum and the integral, by making the change of variables $y = x/h$ in the integral and writing $\alpha = a/h$; then we get

$$I(h) \;=\; \int_{-\infty}^{\infty} h\,\zeta(yh)\log(h^2(y^2+\alpha^2))\,dy \;=\; I_1(h) + I_2(h) \tag{41}$$

where

$$I_1(h) = \int_{-\infty}^{\infty} h\,\zeta(yh)\log(h^2)\,dy \tag{42}$$

and

$$I_2(h) = \int_{-\infty}^{\infty} h\,\zeta(yh)\log(y^2+\alpha^2)\,dy \tag{43}$$

Meanwhile, a similar (except simpler, since without a change of variables) procedure applied to the sum gives

$$S(h) \;=\; S_1(h) + S_2(h) \tag{44}$$

where

$$S_1(h) \;=\; \sum_{j=-\infty}^{\infty} \Big(h\,\zeta(jh)\,\log(h^2)\Big) \tag{45}$$

and

$$S_2(h) = \sum_{j=-\infty}^{\infty} h\,\zeta(jh)\log(j^2+\alpha^2) \tag{46}$$

Now we intend to approximate the respective differences $S_1(h) - I_1(h)$ and $S_2(h) - I_2(h)$. The former is simple, since we notice that

$$S_1(h) - I_1(h) \;=\; 2\log h \left(\sum_{j=-\infty}^{\infty} h\zeta(jh) - \int_{-\infty}^{\infty} \zeta(yh)\,h\,dy \right) \;=\; 2\log h \left(\sum_{j=-\infty}^{\infty} h\zeta(jh) - \int_{-\infty}^{\infty} \zeta(x)\,dx \right) \tag{47}$$

This integrand is a smooth function with all derivatives 0 at infinity; therefore the Euler-Maclaurin summation formula holds, which tells us that the sum in the previous equation approximates the integral in that equation to any degree in h that we care to choose. We apply it to degree 4 to give (for some constant C_1

$$|S_1(h) - I_1(h)| \;\le\; 2\log h(C_1 h^4) \;\le\; 2C_1 h^3 \tag{48}$$

Now turning to the difference $S_2(h) - I_2(h)$, we extract a factor of h from I_2 and S_2, setting (for $h > 0$):

$$\mathcal{I}_2(h) \;=\; \int_{-\infty}^{\infty} \zeta(yh)\log(y^2+\alpha^2)\,dy \;=\; I_2(h)/h \tag{49}$$

and

$$\mathcal{S}_2(h) \;=\; \sum_{j=-\infty}^{\infty} \zeta(jh)\log(j^2+\alpha^2) \;=\; S_2(h)/h \tag{50}$$

Now, we fix α, so that the ratio a/h remains constant, and notice that all of these expressions are differentiable with respect to h at any $h > 0$, since ζ is infinitely differentiable and the rest of each function is independent of h. Then differentiating in (49) and (50) gives

$$\mathcal{I}_2'(h) \;=\; \int_{-\infty}^{\infty} y\,\zeta'(yh)\log(y^2+\alpha^2)\,dy \;=\; 1/h^2 \int_{-\infty}^{\infty} x\,\zeta'(x)\left(\log(x^2+a^2) - \log(h^2)\right)\,dx \tag{51}$$

and

$$\mathcal{S}_2'(h) \;=\; \sum_{j=-\infty}^{\infty} j\,\zeta'(jh)\log(j^2+\alpha^2) \;=\; 1/h^2 \sum_{j=-\infty}^{\infty} h\,jh\,\zeta'(jh)\left(\log((jh)^2+a^2) - \log(h^2)\right) \tag{52}$$

But ζ' is an infinitely differentiable function whose derivatives are all zero at infinity, so the Euler-Maclaurin summation formula tells us that for some constant C_2

$$\left| \int_{-\infty}^{\infty} x\,\zeta'(x)\log(x^2+a^2)\,dx - \sum_{j=-\infty}^{\infty} h\,jh\,\zeta'(jh)\log((jh)^2+a^2) \right| \;\le\; C_2 h^3 \tag{53}$$

Moreover, C_2 is independent of a, because the function $\zeta'(x)\log(x^2+a^2)$ and its first three derivatives are uniformly bounded in a for $0 < a < 1$: because ζ is constant on $[-c, c]$, $[b, \infty]$ and $[-\infty, -b]$, $\zeta' \equiv 0$ on these same intervals, as do all its derivatives, and it is easy to see that the function is uniformly bounded in a on $[c, b]$ and $[-b, -c]$.

Similarly, for some C_3,

$$\left| \int_{-\infty}^{\infty} x\,\zeta'(x)\log(h^2)\,dx - \sum_{j=-\infty}^{\infty} h\,jh\,\zeta'(jh)\log(h^2) \right| \leq C_3 h^3 \qquad (54)$$

And C_3 is clearly independent of a as well.

Putting these together in (51) and (52) gives

$$|\mathcal{I}_2'(h) - \mathcal{S}_2'(h)| \leq (C_2 + C_3)h \qquad (55)$$

Thus we have

$$\lim_{h\to 0^+} (\mathcal{I}_2'(h) - \mathcal{S}_2'(h)) = 0 \qquad (56)$$

And so by Lemma 5, we have $\lim_{h\to 0^+} (\mathcal{I}_2(h) - \mathcal{S}_2(h)) = L$ for some constant L, which gives us the asymptotic approximation

$$I_2(h) - S_2(h) \sim L\,h \text{ as } h \to 0^+ \qquad (57)$$

To find a more precise value for $S_2(h) - I_2(h)$, we use the differentiability (in h) of $\mathcal{S}_2(h)$ and $\mathcal{I}_2(h)$, together with the fundamental theorem of calculus, to write:

$$\mathcal{I}_2(h) - \mathcal{S}_2(h) = \lim_{b\to 0^+} \mathcal{I}_2(b) - \mathcal{S}_2(b) + \int_b^h (\mathcal{I}_2'(x) - \mathcal{S}_2'(x))\;dx = L + \int_0^h (\mathcal{I}_2'(x) - \mathcal{S}_2'(x))\;dx \qquad (58)$$

Now from (55) we have a bound for this integrand, which gives us

$$|\mathcal{I}_2(h) - \mathcal{S}_2(h) - L| \leq \int_0^h (C_2 + C_3)x\,dx \leq (C_2 + C_3)h^2 \qquad (59)$$

Multiplying through by h then gives

$$|I_2(h) - S_2(h) - Lh| \leq (C_2 + C_3)h^3 \qquad (60)$$

And from this taken together with (48), if we combine all of our constants into a single C, we get

$$|I - S - Lh| \leq Ch^3 \qquad (61)$$

This concludes the proof of Lemma 6.

Lemma 7 *Let ζ be any smooth cutoff function as defined above, and set*

$$I = \int_{-\infty}^{\infty} x^2\zeta(x)\log(x^2 + a^2)\,dx \qquad (62)$$

Given any small positive number h, we define also

$$S(h) = \sum_{j=-\infty}^{\infty} \left(h(jh)^2\zeta(jh)\log((jh)^2 + a^2) \right) \qquad (63)$$

Then there is some constant C such that $|I - S| \leq Ch^3$.

PROOF: As we did in the proof of Lemma 6, we make the change of variables $y = x/h$ in the integral and write $\alpha = a/h$ to get

$$I(h) \;=\; \int_{-\infty}^{\infty} h^3 y^2\, \zeta(yh) \log(h^2(y^2+\alpha^2))\, dy \;=\; I_1(h) + I_2(h) \tag{64}$$

where

$$I_1(h) = \int_{-\infty}^{\infty} 2h^3 y^2\, \zeta(yh) \log h\, dy \tag{65}$$

and

$$I_2(h) = \int_{-\infty}^{\infty} h^3 y^2\, \zeta(yh) \log(y^2+\alpha^2)\, dy \tag{66}$$

Similarly, we write

$$S(h) \;=\; S_1(h) + S_2(h) \tag{67}$$

where

$$S_1(h) \;=\; \sum_{j=-\infty}^{\infty} \Big(2h^3 j^2\, \zeta(jh)\, \log h \Big) \tag{68}$$

and

$$S_2(h) = \sum_{j=-\infty}^{\infty} h^3 j^2\, \zeta(jh) \log(j^2+\alpha^2) \tag{69}$$

Just as we saw earlier, $S_1(h)$ has no singularity, so it approximates $I_1(h)$ to $O(h^3)$ accuracy.

Then turning to $S_2(h)$ and $S_1(h)$, extracting a factor of h^3 gives

$$\mathcal{I}_2(h) = \int_{-\infty}^{\infty} y^2\, \zeta(yh) \log(y^2+\alpha^2)\, dy \tag{70}$$

and

$$\mathcal{S}_2(h) = \sum_{j=-\infty}^{\infty} j^2\, \zeta(jh) \log(j^2+\alpha^2) \tag{71}$$

As before, we fix α and differentiate in h to give

$$\mathcal{I}_2'(h) \;=\; \int_{-\infty}^{\infty} y^3\, \zeta'(yh) \log(y^2+\alpha^2)\, dy \;=\; 1/h^4 \int_{-\infty}^{\infty} x^3\, \zeta'(x) \Big(\log(x^2+a^2) - \log(h^2) \Big)\, dx \tag{72}$$

and

$$\mathcal{S}_2'(h) \;=\; \sum_{j=-\infty}^{\infty} j^3\, \zeta'(jh) \log(j^2+\alpha^2) \;=\; 1/h^4 \sum_{j=-\infty}^{\infty} h\,(jh)^3\, \zeta'(jh) \Big(\log((jh)^2+a^2) - \log(h^2) \Big) \tag{73}$$

We recognize this sum as an approximation to the integral of (72), where the near-singularity once again vanishes because of the fact that $\zeta' \equiv 0$ at 0. Thus we have

$$\lim_{h\to 0^+} \left(\mathcal{I}_2'(h) - \mathcal{S}_2'(h) \right) = 0 \tag{74}$$

And so by Lemma 5, we have $\lim_{h\to 0^+} \left(\mathcal{I}_2(h) - \mathcal{S}_2(h) \right) \;=\; C_0$ for some constant C_0, which gives us the asymptotic approximation

$$I_2(h) - S_2(h) \;\sim\; C_0\, h^3 \text{ as } h \to 0^+ \tag{75}$$

which in turn implies that there is some constant C such that

$$|I_2(h) - S_2(h)| \le Ch^3 \tag{76}$$

This concludes the proof of Lemma 7.

We now turn to the case of a general C^4 function, which we state via another lemma:

Lemma 8 *Let ζ be any smooth cutoff function as defined above, and suppose f is a C^4 function on $[-1, 1]$. Given any small positive number h, we define*

$$I(h) = \int_{-1}^{1} f(x)\zeta(x)\log(x^2 + a^2)dx \tag{77}$$

$$S(h) = \sum_{j=-\infty}^{\infty} \Big(h\, f(jh)\zeta(jh)\log((jh)^2 + a^2)\Big) \tag{78}$$

Then there is some constant C independent of h and a and some constant L dependent only on the ratio a/h and independent of f such that

$$|I - S + Lf(0)h| \le Ch^3 \tag{79}$$

Proof: Since f is C^4, we can write a Taylor expansion for f of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + g(x) \tag{80}$$

where g is a C^4 function with the property that $g(0) = g'(0) = g''(0) = g^{(3)}(0) = 0$, and $a_0 = f(0)$, just as we did at the end of the proof of Theorem 1. The rest of this proof then proceeds similarly:

We can write

$$I(h) - S(h) = a_0 R_0(h) + a_1 R_1(h) + a_2 R_2(h) + a_3 R_3(h) + R_g(h) \tag{81}$$

where for $0 \le i \le 3$,

$$R_i(h) = \int_{-\infty}^{\infty} x^i \zeta(x)\log(x^2 + a^2)\,dx - \sum_{j=-\infty}^{\infty} \Big(h(jh)^i \zeta(jh)\log((jh)^2 + a^2)\Big) \tag{82}$$

and

$$R_g(h) = \int_{-\infty}^{\infty} g(x)\zeta(x)\log(x^2 + a^2)\,dx - \sum_{j=-\infty}^{\infty} \Big(h\, g(jh)\zeta(jh)\log((jh)^2 + a^2)\Big) \tag{83}$$

From Lemma 6, we know that there are some constants C_0 and L, where C_0 is independent of a or h and L is independent of f (because there is no f in the statement of the lemma), such that $|R_0(h) + Lh| \le C_0 h^3$, and similarly from Lemma 7, we know that there is some constant C_2 such that $|R_2(h)| \le C_2 h^3$. Also, because ζ is an even function, $x^i \zeta(x)\log(x^2 + a^2)$ is an odd function when i is odd, so both the integral of this function and the finite-sum approximation to it will be zero, giving $R_1(h) = R_3(h) = 0$.

Moreover, we know that the first three derivatives of $g(x)\zeta(x)\log(x^2+a^2)$ exist and are bounded, independently of a, because of the fact that $g(0) = g'(0) = g''(0) = g^{(3)}(0) = 0$, which does away with the near-singularity at the origin. So the Euler-Maclaurin summation formula applies to give third-order accuracy: there is some constant C_g such that $|R_g(h)| \leq C_g h^3$.

Thus, we have established that

$$\begin{aligned} |I(h) - S(h) + a_0 Lh| \;&\leq\; |a_0\,(R_0(h) + Lh)| + |a_1 R_1(h)| + |a_2 R_2(h)| + |a_3 R_3(h)| + |R_g(h)| \\ &\leq\; |a_0| C_0 h^3 + |a_2| C_2 h^3 + C_g h^3 \qquad (84) \end{aligned}$$

And by setting $C = |a_0|C_0 + |a_2|C_2 + C_g$, the statement of the lemma follows.

We now turn to a final lemma to extend the previous theorem so that we do not need the stipulation that the function is a multiple of some cutoff function ζ:

Lemma 9 *Suppose f is a C^4 function on $[-1,1]$. Given any large positive integer N, with $h = 1/N$, we define I and S as in the statement of Theorem 2. Then there is some constant C independent of h and a and some constant L dependent only on the ratio a/h and independent of f such that*

$$|I - S + Lf(0)h| \leq Ch^3 \qquad (85)$$

Proof: Choose some smooth cutoff function ζ of the kind defined above (so that its entire support is contained within $[-1,1]$) and write

$$I \;=\; I_A(h) + I_B(h) \qquad (86)$$

where

$$I_A(h) \;=\; \int_{-1}^{1} f(x)\zeta(x)\log(x^2 + a^2)dx \qquad (87)$$

and

$$I_B(h) \;=\; \int_{-1}^{1} f(x)(1 - \zeta(x))\log(x^2 + a^2)dx \qquad (88)$$

Similarly, write

$$S \;=\; S_A(h) + S_B(h) \qquad (89)$$

where

$$S_A(h) \;=\; \sum_{j=-N}^{N} \Big(f(jh)\zeta(jh)\log((jh)^2 + a^2) w_j\, h \Big) \qquad (90)$$

$$S_B(h) \;=\; \sum_{j=-N}^{N} \Big(f(jh)(1 - \zeta(jh))\log((jh)^2 + a^2) w_j\, h \Big) \qquad (91)$$

Since we chose ζ to be nonzero only within $[-1,1]$, we can change the limits of integration and summation in I_A and S_A to be $\pm\infty$ without changing the value of these, so that from Lemma 8 we can say that there is some constant C_A so that

$$|I_A(h) - S_A(h) + Lhf(0)| \;\leq\; C_A h^3 \qquad (92)$$

And in I_B and S_B, the integrand $f(x)(1-\zeta(x))\log(x^2+a^2)$ and all of its derivatives are exactly 0 in a neighborhood of 0 (whose size does not depend on h or a) because of the fact that $\zeta \equiv 1$ in such a neighborhood. Thus there is no near-singularity to worry about; all derivatives are bounded independently of h and a, and so we can apply a standard rule like that of Lemma 1 to give third-order accuracy:

$$|I_B(h) - S_B(h)| \le C_B h^3 \tag{93}$$

We could have gotten higher-order accuracy had we chosen a different set of weights w_j, but we already have a third-order error in (92), so this is sufficient for our purposes. If we did want a higher degree of accuracy, we could get fifth-order accuracy or higher (assuming sufficiently smooth f) by choosing weights appropriately and by introducing one or more additional correction term based on $f''(0)$ and subsequent derivatives at 0.

Here, however, third-order accuracy is sufficient, and so we combine equations (92) and (93) to give

$$|I - S + Lhf(0)| \le |I_A(h) - S_A(h) + Lhf(0)| + |I_B(h) - S_B(h)| \le C_A h^3 + C_B h^3 \tag{94}$$

so that writing $C = C_A + C_B$ concludes the proof of Lemma 9.

We have thus nearly proved Theorem 2, except that we still need to show that the constant L is given by the value that we claimed it was, namely $\log(1 - e^{-2\pi a/h})$.

Proof of Theorem 2: We know that L is independent of our choice of function f, so we can choose f to be anything we would like; we'll pick the simplest possible option, $f \equiv 1$.

To establish the value of L, we will need to refer to the Poisson Summation Formula, which we state from [[3], p. 138]; this is a consequence of the Euler-Maclaurin Summation Formula that applies to any function g that has a Fourier transform:

$$\int_0^p f(x)\,dx = \delta\left(\tfrac{1}{2}(g(0) + g(p)) + \sum_{k=1}^{n-1} g(k\delta)\right) - \sum_{k=-\infty,\, k\neq 0}^{\infty} \tilde{g}(2k\pi/\delta) \tag{95}$$

where $\delta = p/n$ and $\tilde{g}(t) = \int_{-p}^{p} f(x)\cos(tx)\,dx$.

We will apply (95), taking $x = y$, $g(y) = \log(y^2 + \alpha^2)$, $p = 1/h = N$, $\delta = 1$, and $n = p = N$, to give

$$\int_0^N \log(y^2+\alpha^2)\,dy = \left(\tfrac{1}{2}(2\log(\alpha) + \log(N^2+\alpha^2)) + \sum_{k=1}^{N-1} \log(k^2+\alpha^2)\right) - \sum_{k=-\infty,k\neq 0}^{\infty} \tilde{g}(2k\pi) \tag{96}$$

In this case, we have

$$\tilde{g}(2k\pi) = \int_{-N}^{N} \log(y^2+\alpha^2)\cos(2k\pi y)\,dy \tag{97}$$

so that integrating by parts gives first

$$\tilde{g}(2k\pi) = \left[\frac{1}{2k\pi}\sin(2k\pi y)\log(y^2+\alpha^2)\right]_{-N}^{N} - \frac{1}{2k\pi}\int_{-N}^{N} \frac{2y}{y^2+\alpha^2}\sin(2k\pi y)\,dy \tag{98}$$

and then

$$\tilde{g}(2k\pi) = \left[\frac{1}{2k\pi}\sin(2k\pi y)\log(y^2+\alpha^2)\right]_{-N}^{N} + \left[\frac{1}{4k^2\pi^2}\left(\frac{2y}{y^2+\alpha^2}\right)\cos(2k\pi y)\right]_{-N}^{N}$$

$$-\frac{1}{4k^2\pi^2}\int_{-N}^{N}\frac{2(\alpha^2-y^2)}{(y^2+\alpha^2)^2}\cos(2k\pi y)\,dy \tag{99}$$

We then approximate this last integral by the integral of the same integrand taken over the entire real line, writing

$$\int_{-N}^{N}\frac{2(\alpha^2-y^2)}{(y^2+\alpha^2)^2}\cos(2k\pi y)\,dy \;=\; \int_{-\infty}^{\infty}\frac{2(\alpha^2-y^2)}{(y^2+\alpha^2)^2}\cos(2k\pi y)\,dy + O\left(\frac{1}{N}\right) \tag{100}$$

since

$$\left|\int_{N}^{\infty}\frac{2(\alpha^2-y^2)}{(y^2+\alpha^2)^2}\cos(2k\pi y)\,dy\right| \;\le\; \max|\cos(2k\pi y)|\,\max\left|\frac{\alpha^2-y^2}{\alpha^2+y^2}\right|\left|\int_{N}^{\infty}\frac{2}{\alpha^2+y^2}\,dy\right|$$

$$< 1\cdot 1\int_{N}^{\infty}\frac{2}{y^2}\,dy \;=\; \frac{2}{N} \tag{101}$$

and the same is true of the integral over $(-\infty,-N)$.

Then it is easy to show that the integral on the right-hand side of (100), which is really just a Fourier transform, is given by

$$\int_{-\infty}^{\infty}\frac{2(\alpha^2-y^2)}{(y^2+\alpha^2)^2}\cos(2k\pi y)\,dy \;=\; 4|k|\pi^2 e^{-2\pi\alpha|k|} \tag{102}$$

so that (99) becomes

$$\tilde{g}(2k\pi) \;=\; \left[\frac{1}{2k\pi}\sin(2k\pi y)\log(y^2+\alpha^2)\right]_{-N}^{N} + \left[\frac{1}{4k^2\pi^2}\left(\frac{2y}{y^2+\alpha^2}\right)\cos(2k\pi y)\right]_{-N}^{N}$$

$$-\frac{1}{4k^2\pi^2}\left(4|k|\pi^2 e^{-2\pi\alpha|k|} + O\left(\frac{1}{N}\right)\right) \tag{103}$$

The fact that N and k are both integers allows us to simplify this further:

$$\tilde{g}(2k\pi) \;=\; 0 + \frac{2}{4k^2\pi^2}\frac{2N}{N^2+\alpha^2} - \frac{1}{|k|}e^{-2\pi\alpha|k|} + O\left(\frac{1}{Nk^2}\right) \;=\; -\frac{1}{|k|}e^{-2\pi\alpha|k|} + O\left(\frac{1}{Nk^2}\right) \tag{104}$$

Then the second sum in (96) becomes:

$$\sum_{k=-\infty,\,k\neq 0}^{\infty}\tilde{g}(2k\pi) \;=\; \sum_{k=-\infty,\,k\neq 0}^{\infty}\left(-\frac{1}{|k|}e^{-2\pi\alpha|k|} + O\left(\frac{1}{Nk^2}\right)\right) \;=\; \sum_{k=1}^{\infty} -\frac{2}{k}e^{-2\pi\alpha k} + O\left(\frac{1}{N}\right) \tag{105}$$

where we have used the fact that $\sum_{k=0}^{\infty} 1/k^2$ is a constant.

In general, for any real number b with $|b| < 1$, we can write

$$\sum_{n=1}^{\infty} b^n/n \;=\; \sum_{n=1}^{\infty}\int_0^b \beta^{n-1}\,d\beta \tag{106}$$

Then we can interchange the order of integration and summation to give

$$\sum_{n=1}^{\infty} b^n/n \;=\; \int_0^b\left(\sum_{n=1}^{\infty}\beta^{n-1}\right)d\beta \;=\; \int_0^b (1-\beta)^{-1}\,d\beta \;=\; -\log(1-\beta)|_0^b \;=\; -\log(1-b) \tag{107}$$

Applying this result to (105) with $n = k$ and $b = e^{-2\pi\alpha}$, we get

$$\sum_{k=-\infty,\, k\neq 0}^{\infty} \tilde{g}(2k\pi) \;=\; 2\log(1 - e^{-2\pi\alpha}) + O\left(\frac{1}{N}\right) \tag{108}$$

And substituting this expression back into (96), we get

$$\int_0^N \log(y^2 + \alpha^2)\, dy \;=\; \sum_{k=0}^{N} t_k \log(k^2 + \alpha^2) - 2\log(1 - e^{-2\pi\alpha}) + O\left(\frac{1}{N}\right) \tag{109}$$

where t_k are the weights of the trapezoid rule; $t_k = 1$ except that $t_0 = t_N = \frac{1}{2}$.

Now, we can multiply both sides of this equation by h and make a change of variables $x = hy$ to produce

$$\int_0^1 \Big(\log(x^2 + a^2) - 2\log h\Big)\; dx \;=\; \sum_{j=0}^{N} t_j \Big(\log((jh)^2 + a^2) - 2\log(h)\Big)\, h - 2h\log(1 - e^{-2\pi\alpha}) + O(h^2) \tag{110}$$

Now the trapezoid rule tells us that for some constant C,

$$\left| \int_0^1 2\log h\, dx - \sum_{j=0}^{N} 2t_j h \log h \right| \;\leq\; \log h(Ch^2) \tag{111}$$

so that (110) becomes

$$\int_0^1 \log(x^2 + a^2)\, dx \;=\; \sum_{j=0}^{N} t_j \log((jh)^2 + a^2) h - 2h\log(1 - e^{-2\pi\alpha}) + O(h^2 \log h) \tag{112}$$

Now if we could replace the trapezoidal weights t_j with the third-order weights w_j that we are using here, we would be finished. But this is simple, since these differ only at the endpoints of $[-1, 1]$; more exactly,

$$\sum_{j=0}^{N} t_j \log((jh)^2 + a^2)h - \sum_{j=0}^{N} w_j \log((jh)^2 + a^2)h \;=\; \frac{h}{6}\Big(\log(1 + a^2) - \log((1-h)^2 + a^2)\Big) \;=\; O(h^2) \tag{113}$$

so that we can replace (112) by

$$\int_0^1 \log(x^2 + a^2)\, dx \;=\; \sum_{j=0}^{N} w_j \log((jh)^2 + a^2)h - 2h\log(1 - e^{-2\pi\alpha}) + O(h^2 \log h) \tag{114}$$

Dividing through by h gives

$$L = -2\log(1 - e^{-2\pi\alpha}) + O(h \log h) \tag{115}$$

These are both constants so the error term must be zero, giving

$$L = -2\log(1 - e^{-2\pi a/h}) \tag{116}$$

This concludes the proof of Theorem 2.

5 Examples

Here we display the results of a naive attempt to approximate a few sample singular or nearly singular integrals via the rule of Lemma 1, and then the results with the corrected approximation of Theorem 1 or 2.

First, we look at two singular integrals of the form

$$I = \int_{-1}^{1} f(x) \log |x| \, dx \tag{117}$$

We observe the uncorrected error $S - I$ where

$$S = \sum_{j=-N,\, j\neq 0}^{N} f(jh) \log |jh| w_j h \tag{118}$$

And we consider the corrected error $S - I + h \log\left(\frac{h}{2\pi}\right) f(0)$ from Theorem 1.

Example A1: $f(x) = 1$, $I = 2$

N	*Uncorrected Error*	*Corrected Error*	*Corrected Error*$/h^3$
10	4.1×10^{-1}	-9.0×10^{-6}	-0.0899
20	2.4×10^{-1}	-1.1×10^{-6}	-0.0865
40	1.4×10^{-1}	-1.3×10^{-7}	-0.0849
80	7.8×10^{-2}	-1.6×10^{-8}	-0.0841

Example A2: $f(x) = \cos(x)$, $I = -1.89217\ldots$

N	*Uncorrected Error*	*Corrected Error*	*Corrected Error*$/h^3$
10	4.1×10^{-1}	-2.2×10^{-5}	-0.2221
20	2.4×10^{-1}	-2.7×10^{-6}	-0.2188
40	1.4×10^{-1}	-3.4×10^{-7}	-0.2172
80	7.8×10^{-2}	-4.2×10^{-8}	-0.2165

Now, we consider (for different values of a) two nearly singular integrals of the form

$$I = \int_{-1}^{1} f(x) \log(x^2 + a^2) \, dx \tag{119}$$

We observe the uncorrected error $S - I$ where

$$S = \sum_{j=-N}^{N} f(jh) \log(j^2h^2 + a^2) w_j h \tag{120}$$

And we consider the corrected error $S - I + h \log\left(1 - e^{-2\pi a/h}\right) f(0)$ from Theorem 2.

Example B1: $f(x) = 1$, $a = 10^{-3}$, $I = -3.99372\ldots$

N	*Uncorrected Error*	*Corrected Error*	*Corrected Error*$/h^3$
10	-5.6×10^{-1}	-1.8×10^{-4}	-0.1798
20	-2.1×10^{-1}	-2.2×10^{-5}	-0.1730
40	-7.5×10^{-2}	-2.7×10^{-6}	-0.1698
80	-2.3×10^{-2}	-3.3×10^{-7}	-0.1682

Example B2: $f(x) = 1$, $a = 10^{-6}$, $I = -3.999994\ldots$

N	*Uncorrected Error*	*Corrected Error*	*Corrected Error*$/h^3$
10	-1.9×10^{0}	-1.8×10^{-4}	-0.1798
20	-9.0×10^{-1}	-2.2×10^{-5}	-0.1730
40	-4.1×10^{-1}	-2.7×10^{-6}	-0.1698
80	-1.9×10^{-1}	-3.3×10^{-7}	-0.1682

Example B3: $f(x) = 1 - x^2$, $a = 10^{-3}$, $I = -3.5492\ldots$

N	*Uncorrected Error*	*Corrected Error*	*Corrected Error*$/h^3$
10	-5.6×10^{-1}	-7.9×10^{-4}	-0.7890
20	-2.1×10^{-1}	-9.9×10^{-5}	-0.7882
40	-7.5×10^{-2}	-1.2×10^{-5}	-0.7870
80	-2.3×10^{-2}	-1.5×10^{-6}	-0.7831

Example B4: $f(x) = 1 - x^2$, $a = 10^{-6}$, $I = -3.55554\ldots$

N	*Uncorrected Error*	*Corrected Error*	*Corrected Error*$/h^3$
10	-1.9×10^{0}	-7.9×10^{-4}	-0.7891
20	-9.0×10^{-1}	-9.9×10^{-5}	-0.7886
40	-4.1×10^{-1}	-1.2×10^{-5}	-0.7885
80	-1.9×10^{-1}	-1.5×10^{-6}	-0.7885

As predicted, the corrected approximation converges to the exact value of the integral I with $O(h^3)$ accuracy, independently of a.

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