# MATH 340 – SPRING 2026 – HOMEWORK 1

Due Thursday, January 15, 2026 at 8am on Gradescope.

You are encouraged to collaborate with other students, but you must write up your solutions individually, without reference to notes from the collaboration. You may not search the internet or ask AI for solutions to the homework problems. Exception: it is fine to use AI to "vibe-code" the programming questions if you want, but make sure you understand what the code is doing and how to modify it.

**Problem 1** (Inclusion–exclusion principle)**.** Let $A_1, A_2, \dots \subseteq \Omega$ be events. Let's try to prove by induction that, for each $k \geq 1$,

$$P(A_1 \cup \cdots \cup A_k) = \sum_{\ell=1}^{k} (-1)^{\ell+1} \sum_{1 \leq j_1 < \cdots < j_\ell \leq k} P(A_{j_1} \cap \cdots \cap A_{j_\ell}). \tag{1}$$

(a) Write down (1) for $k = 1$ and observe that it is trivial. Write down (1) for $k = 2$ and observe that it is equivalent to a result stated in class. Give a direct proof of this result for $k = 2$. Write down (1) for $k = 3$ and draw a picture to illustrate why it should hold in this case. (You don't need to give a formal argument for $k = 3$.)

(b) Suppose that (1) holds for some $k \geq 1$. Prove that it holds for $k + 1$ as well. [*Hint*: write

$$P\left(\bigcup_{j=1}^{k+1} A_j\right) = P\left(\left(\bigcup_{j=1}^{k} A_j\right) \cup A_{k+1}\right)$$

and use the result for $k = 2$. At some point, you will want to break the sum $\sum_{1 \leq j_1 < \cdots < j_\ell \leq k+1}$ into the case when $j_\ell = k + 1$ and the case when $j_\ell \neq k + 1$.]

(c) Conclude by induction that (1) holds for all $k$.

**Solution.**

(a) For $k = 1$, (1) becomes $P(A_1) = P(A_1)$, which is of course trivial. For $k = 2$, (1) is $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$, which we stated in class. To prove it, note that $P(A_1 \cup A_2) = P(A_1 \cup (A_2 \setminus A_1)) = P(A_1) + P(A_2 \setminus A_1)$ since $A_1$ and $A_2 \setminus A_1$ are disjoint. Next, note that $P(A_2) = P(A_1 \cap A_2) + P(A_2 \setminus A_1)$ since $A_2 = (A_1 \cap A_2) \cup (A_2 \setminus A_1)$ and the sets $A_1 \cap A_2$ and $A_2 \setminus A_1$ are disjoint. Combining the last two identities of probabilities gives us $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$. For $k = 3$, (1) becomes $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$.

(b) We have

$$\begin{aligned}
P\left(\bigcup_{j=1}^{k+1} A_j\right) &= P\left(\left(\bigcup_{j=1}^{k} A_j\right) \cup A_{k+1}\right) \\
&= P\left(\bigcup_{j=1}^{k} A_j\right) + P(A_{k+1}) - P\left(\left(\bigcup_{j=1}^{k} A_j\right) \cap A_{k+1}\right) \\
&= P\left(\bigcup_{j=1}^{k} A_j\right) + P(A_{k+1}) - P\left(\bigcup_{j=1}^{k} (A_j \cap A_{k+1})\right)
\end{aligned}$$

by the $k = 2$ case and de Morgan's laws. Then we use the inductive hypothesis to write this as

$$\begin{aligned}
P\left(\bigcup_{j=1}^{k+1} A_j\right) &= \sum_{\ell=1}^{k}(-1)^{\ell+1} \sum_{1\le j_1<\cdots<j_\ell\le k} P\left(A_{j_1}\cap\cdots\cap A_{j_\ell}\right) \\
&\quad + P(A_{k+1}) \\
&\quad - \sum_{\ell=1}^{k}(-1)^{\ell+1} \sum_{1\le j_1<\cdots<j_\ell\le k} P\left((A_{j_1}\cap A_{k+1})\cap\cdots\cap(A_{j_\ell}\cap A_{k+1})\right) \\
&= \sum_{\ell=1}^{k}(-1)^{\ell+1} \sum_{1\le j_1<\cdots<j_\ell\le k} P\left(A_{j_1}\cap\cdots\cap A_{j_\ell}\right) \\
&\quad + P(A_{k+1}) \\
&\quad - \sum_{\ell=1}^{k}(-1)^{\ell+1} \sum_{1\le j_1<\cdots<j_\ell\le k} P\left(A_{j_1}\cap\cdots\cap A_{j_\ell}\cap A_{k+1}\right).
\end{aligned}$$

On the other hand, we can write

$$\begin{aligned}
&\sum_{\ell=1}^{k+1}(-1)^{\ell+1} \sum_{1\le j_1<\cdots<j_\ell\le k+1} P\left(A_{j_1}\cap\cdots\cap A_{j_\ell}\right) \\
&= \sum_{\ell=1}^{k+1}(-1)^{\ell+1}\left(\sum_{\substack{1\le j_1<\cdots<j_\ell\le k+1\\ j_\ell=k+1}} P\left(A_{j_1}\cap\cdots\cap A_{j_\ell}\right) + \sum_{\substack{1\le j_1<\cdots<j_\ell\le k+1\\ j_\ell\le k}} P\left(A_{j_1}\cap\cdots\cap A_{j_\ell}\right)\right) \\
&= \sum_{\ell=1}^{k+1}(-1)^{\ell+1}\left(\sum_{1\le j_1<\cdots<j_{\ell-1}\le k} P\left(A_{j_1}\cap\cdots\cap A_{j_{\ell-1}}\cap A_{k+1}\right) + \sum_{1\le j_1<\cdots<j_\ell\le k} P\left(A_{j_1}\cap\cdots\cap A_{j_\ell}\right)\right) \\
&= \sum_{\ell=0}^{k}(-1)^{\ell} \sum_{1\le j_1<\cdots<j_\ell\le k} P\left(A_{j_1}\cap\cdots\cap A_{j_\ell}\cap A_{k+1}\right) + \sum_{\ell=1}^{k+1}(-1)^{\ell+1} \sum_{1\le j_1<\cdots<j_\ell\le k} P\left(A_{j_1}\cap\cdots\cap A_{j_\ell}\right) \\
&= P(A_{k+1}) - \sum_{\ell=1}^{k}(-1)^{\ell+1} \sum_{1\le j_1<\cdots<j_\ell\le k} P\left(A_{j_1}\cap\cdots\cap A_{j_\ell}\cap A_{k+1}\right) \\
&\quad + \sum_{\ell=1}^{k}(-1)^{\ell+1} \sum_{1\le j_1<\cdots<j_\ell\le k} P\left(A_{j_1}\cap\cdots\cap A_{j_\ell}\right).
\end{aligned}$$

In the first identity we use that any tuple $(j_1,\dots,j_\ell)$ with $1\le j_1<\cdots<j_\ell\le k+1$ either has $j_\ell = k+1$ or $j_\ell\le k$. In the last identity we used that in the last sum on the r.h.s., the term $\ell = k+1$ does not contribute because there are no tuples $(j_1,\dots,j_{k+1})$ such that $1\le j_1<j_2<\cdots<j_{k+1}\le k$. The last terms in the last two displays match, and so we conclude (1) for $k+1$.

(c) We proved the base case $k = 1$ in part (a) and the inductive step in part (b). Therefore, the conclusion holds by induction.

**Problem 2.**

(a) Prove, using mathematical induction, that if $E_1, E_2, \ldots, E_n$ are events, then

$$\mathbb{P}\left(\bigcap_{j=1}^{n} E_j\right) \geq \sum_{j=1}^{n} \mathbb{P}(E_j) - (n-1). \tag{2}$$

(b) State and prove a (nontrivial) condition on the events $E_1, \ldots, E_n$ for equality to hold in (2).

**Solution.**

(a) The base case $n = 1$ is trivial. Now suppose that (2) holds for $n = k$, i.e. that $\mathbb{P}\left(\bigcap_{j=1}^{k} E_j\right) \geq \sum_{j=1}^{k} \mathbb{P}(E_j) - (k-1)$, and we'll try to prove that it holds for $n = k+1$. For this we can write

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{j=1}^{k+1} E_j\right) &= 1 - \mathbb{P}\left(\left(\bigcap_{j=1}^{k+1} E_j\right)^{\mathrm{c}}\right) = 1 - \mathbb{P}\left(\bigcup_{j=1}^{k+1} E_j^{\mathrm{c}}\right) = 1 - \mathbb{P}\left(\left(\bigcup_{j=1}^{k} E_j^{\mathrm{c}}\right) \cup E_{k+1}^{\mathrm{c}}\right) \\
&= 1 - \mathbb{P}\left(\bigcup_{j=1}^{k} E_j^{\mathrm{c}}\right) - \mathbb{P}(E_{k+1}^{\mathrm{c}}) + \mathbb{P}\left(\left(\bigcup_{j=1}^{k} E_j^{\mathrm{c}}\right) \cap E_{k+1}^{\mathrm{c}}\right) \\
&= 1 - \mathbb{P}\left(\left(\bigcap_{j=1}^{k} E_j\right)^{\mathrm{c}}\right) - (1 - \mathbb{P}(E_{k+1})) + \mathbb{P}\left(\left(\bigcup_{j=1}^{k} E_j^{\mathrm{c}}\right) \cap E_{k+1}^{\mathrm{c}}\right) \\
&= \mathbb{P}\left(\bigcap_{j=1}^{k} E_j\right) + \mathbb{P}(E_{k+1}) - 1 + \mathbb{P}\left(\left(\bigcup_{j=1}^{k} E_j^{\mathrm{c}}\right) \cap E_{k+1}^{\mathrm{c}}\right) \\
&\geq \sum_{j=1}^{k} \mathbb{P}(E_j) - (k-1) + \mathbb{P}(E_{k+1}) - 1 \\
&= \sum_{j=1}^{k+1} \mathbb{P}(E_j) - ((k+1) - 1),
\end{aligned}$$

where we used de Morgan's laws in the identities, and then in the inequality we used the inductive hypothesis and the fact that $\mathbb{P}\left(\left(\bigcup_{j=1}^{k} E_j^{\mathrm{c}}\right) \cap E_{k+1}^{\mathrm{c}}\right) \geq 0$. This is what we needed to show.

(b) Equality holds if $E_i \cup E_j = \Omega$ for each $i, j$. We can prove this again by induction in a very similar way. The base case $n = 1$ is again trivial. For the inductive step, we again write as above

$$\mathbb{P}\left(\bigcap_{j=1}^{k+1} E_j\right) = \mathbb{P}\left(\bigcap_{j=1}^{k} E_j\right) + \mathbb{P}(E_{k+1}) - 1 + \mathbb{P}\left(\left(\bigcup_{j=1}^{k} E_j^{\mathrm{c}}\right) \cap E_{k+1}^{\mathrm{c}}\right).$$

We can write the last probability as $\mathbb{P}\left(\left(\bigcup_{j=1}^{k} E_j^{\mathrm{c}}\right) \cap E_{k+1}^{\mathrm{c}}\right) = \mathbb{P}\left(\bigcup_{j=1}^{k} (E_j^{\mathrm{c}} \cap E_{k+1}^{\mathrm{c}})\right) = \mathbb{P}\left(\bigcup_{j=1}^{k} (E_j \cup E_{k+1})^{\mathrm{c}}\right) = \mathbb{P}\left(\bigcup_{j=1}^{k} \Omega^{\mathrm{c}}\right) = \varnothing$, and hence the last probability in this case is 0. So we can write using the inductive hypothesis that

$$\mathbb{P}\left(\bigcap_{j=1}^{k+1} E_j\right) = \mathbb{P}\left(\bigcap_{j=1}^{k} E_j\right) + \mathbb{P}(E_{k+1}) - 1 = \sum_{j=1}^{k} \mathbb{P}(E_j) - (k-1) + \mathbb{P}(E_{k+1}) - 1 = \sum_{j=1}^{k+1} \mathbb{P}(E_j) - ((k+1) - 1)$$

and this completes the proof.

**Problem 3.** Suppose that $n$ people are getting on an airplane, but everyone is feeling rather chaotic and sits in a uniformly random unoccupied seat (as opposed to sitting in their assigned seat, although they may happen to sit in their assigned seat). The point of this problem is to compute the probability that *no one* sits in their assigned seat.

(a) Let $E_j$ be the event that the $j$th person sits in their own seat. For $j_1 < j_2 < \cdots < j_\ell$, compute $P(E_{j_1} \cap \cdots \cap E_{j_\ell})$.

(b) Use the previous part and (1) to compute $P(E_1 \cup \cdots \cup E_n)$ as a sum of $n$ terms.

(c) Compute

$$\lim_{n\to\infty} \left(1 - P(E_1 \cup \cdots \cup E_n)\right).$$

[*Hint*: recognize the sum you computed in the previous part as a Taylor series.]

(d) Write a simulation code to run many (try 10000 or 100000) trials of this experiment for reasonably large $n$ (try $n = 100$ or $n = 1000$) and compute the fraction of the trials in which no one ends up in their assigned seat. [If you use Python, the random.shuffle function will give you a uniform random permutation. It is also fine to "vibe-code" this part if you are not comfortable with programming.] Is the number you get close to the limit you computed in the previous part? [It should be reasonably close if you choose $n$ large enough and run enough trials.] Attach your code and results to your submission.

**Solution.**

(a) Since the labeling of the seats is irrelevant, we see that every permutation of $\{1, \ldots, n\}$ is equally likely, i.e. the assignment of people to seats is distributed uniformly on the set of all $n!$ permutations of $\{1, \ldots, n\}$. The number of such permutations such that any given $\ell$ people are sitting in the correct seat is the number of ways of assigning all of the remaining $n - \ell$ people to arbitary seats, i.e. $(n - \ell)!$. So the probability of this event is $(n - \ell)!/n!$.

(b) By Problem 1, we have

$$\begin{aligned} P\left(E_1 \cup \cdots \cup E_n\right) &= \sum_{\ell=1}^{n} (-1)^{\ell+1} \sum_{1\le j_1<\cdots<j_\ell\le n} P\left(E_{j_1} \cap \cdots \cap E_{j_\ell}\right) \\ &= \sum_{\ell=1}^{n} (-1)^{\ell+1} \sum_{1\le j_1<\cdots<j_\ell\le n} \frac{(n-\ell)!}{n!}. \end{aligned}$$

Since there are $\binom{n}{\ell}$ ways of selecting an increasing sequence $1 \le j_1 < \cdots < j_\ell \le n$, this becomes

$$P\left(E_1 \cup \cdots \cup E_n\right) = \sum_{\ell=1}^{n} (-1)^{\ell+1} \binom{n}{\ell} \frac{(n-\ell)!}{n!} = \sum_{\ell=1}^{n} \frac{(-1)^{\ell+1}}{\ell!}.$$

(c) We can write

$$1 - P\left(E_1 \cup \cdots \cup E_n\right) = \sum_{\ell=0}^{n} \frac{(-1)^\ell}{\ell!} \xrightarrow[n\to\infty]{} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} = \mathrm{e}^{-1}.$$