MATH 340 – SPRING 2026 – HOMEWORK 2

Due Thursday, January 22, 2026 at 8am on Gradescope. You must justify all of your answers for full credit.

You are encouraged to collaborate with other students, but you must write up your solutions individually, without reference to notes from the collaboration. You may not search the internet or ask AI for solutions to the homework problems. Exception: it is fine to use AI to "vibe-code" the programming questions if you want, but make sure you understand what the code is doing and how to modify it.

Problem 1 (based on Meester, Exercise 1.7.14)**.** There is a single-elimination tennis tournament where each of the 2^n players is seeded at random. Assume that all of the players are evenly matched and the games are independent (and independent of the seedings as well), so each player wins each game with probability $1/2$. Given two players in the tournament, what is the probability that they play one another? Justify your answer.

Solution. We condition on how far apart the players are in the starting bracket. Given one player (call him Cole), the probability of the event E_k of another randomly seeded player (call him Chiranjib) first possibly playing Cole in the kth round is given by $(2^k - 2^{k-1})/(2^n - 1)$, since there are $2^n - 1$ possible positions for Chiranjib in the bracket and $2^k - 2^{k-1}$ players in Cole's depth-k subbracket who are not in Cole's depth-$(k-1)$ subbracket. Conditional on the event E_k, the probability that they actually play one another is given by the probability that they both win the $k-1$ games necessary to reach the kth round, which has probability $2^{-2(k-1)}$. So the probability that Cole and Chiranjib play each other is

$$\begin{aligned}\sum_{k=1}^{n} P(\text{play each other} \mid E_k)P(E_k) &= \sum_{k=1}^{n} \frac{(2^k - 2^{k-1})}{2^n - 1} \cdot 2^{-2(k-1)} \\ &= \frac{4}{2^n - 1}\sum_{k=1}^{n}(2^{-k} - 2^{-k-1}) \\ &= \frac{2}{2^n - 1}\sum_{k=1}^{n} 2^{-k} \\ &= \frac{2(1 - 2^{-n})}{2^n - 1} \\ &= 2^{-n+1}.\end{aligned}$$

Problem 2. Recall the hat game discussed in the lecture of January 13 (or see Example 1.4.5 in Meester). The purpose of this problem is to show that no deterministic strategy can win with probability greater than $3/4$. So let us fix some (unspecified) deterministic strategy. (A *deterministic* strategy is one where the players' guesses depend only on the hats that they see, not on any other pieces of information or randomness.)

(a) Let $Y_{\mathrm{A}}, Y_{\mathrm{B}}, Y_{\mathrm{C}}$ be the events that A, B, C make correct guesses, respectively. Similarly, let $N_{\mathrm{A}}, N_{\mathrm{B}}, N_{\mathrm{C}}$ be the events that A, B, C make incorrect guesses, respectively. Argue that $P(Y_{\mathrm{X}}) = P(N_{\mathrm{X}})$ for each $\mathrm{X} \in \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$.

Recall that the players win if at least one of them makes a correct guess, and none of them makes an incorrect guess, i.e. on the event $W := Y_{\mathrm{A}} \cup Y_{\mathrm{B}} \cup Y_{\mathrm{C}} \setminus (N_{\mathrm{A}} \cup N_{\mathrm{B}} \cup N_{\mathrm{C}})$.

(b) Show that if there is some $\mathrm{X} \in \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ such that $P(Y_{\mathrm{X}}) \geq 1/4$, then $P(W) \leq 3/4$.

(c) Show that if, on the other hand, there is no $X \in \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ such that $P(Y_{\mathrm{X}}) \geq 1/4$, then $P(W) \leq 3/4$ as well.

Solution.

(a) Let's consider the case X = A; the other cases are similar. Since A's guess depends only on the hats of B and C, there are disjoint $E_{\mathrm{red}}, E_{\mathrm{blue}}, E_{\mathrm{pass}} \subseteq \Omega' := \{\mathrm{red}, \mathrm{blue}\}^2$ such that $E_{\mathrm{red}} \cup E_{\mathrm{blue}} \cup E_{\mathrm{pass}} = \Omega'$ and A guesses "red", "blue", and "pass" on the events $\{\mathrm{red}, \mathrm{blue}\} \times E_{\mathrm{red}}$, $\{\mathrm{red}, \mathrm{blue}\} \times E_{\mathrm{blue}}$, and $\{\mathrm{red}, \mathrm{blue}\} \times E_{\mathrm{pass}}$, respectively. This means that $Y_{\mathrm{A}} = (\{\mathrm{red}\} \times E_{\mathrm{red}}) \cup (\{\mathrm{blue}\} \times E_{\mathrm{blue}})$ and $N_{\mathrm{A}} = (\{\mathrm{blue}\} \times E_{\mathrm{red}}) \cup (\{\mathrm{red}\} \times E_{\mathrm{blue}})$, and so

$$\begin{aligned} P(Y_{\mathrm{A}}) &= P(\{\mathrm{red}\} \times E_{\mathrm{red}}) + P(\{\mathrm{blue}\} \times E_{\mathrm{blue}}) \\ &= P(\{\mathrm{blue}\} \times E_{\mathrm{red}}) + P(\{\mathrm{red}\} \times E_{\mathrm{blue}}) \\ &= P(N_{\mathrm{A}}) \end{aligned}$$

since the coin tosses are independent.

(b) In this case we have

$$\begin{aligned} P(W) &= P(Y_{\mathrm{A}} \cup Y_{\mathrm{B}} \cup Y_{\mathrm{C}} \setminus (N_{\mathrm{A}} \cup N_{\mathrm{B}} \cup N_{\mathrm{C}})) \\ &\leq P(\Omega \setminus N_{\mathrm{A}}) = 1 - P(N_{\mathrm{A}}) = 1 - P(Y_{\mathrm{A}}) \leq \frac{3}{4}, \end{aligned}$$

where we used that $P(N_{\mathrm{A}}) = P(Y_{\mathrm{A}})$ by the previous part.

(c) In this case we have

$$\begin{aligned} P(W) &= P(Y_{\mathrm{A}} \cup Y_{\mathrm{B}} \cup Y_{\mathrm{C}} \setminus (N_{\mathrm{A}} \cup N_{\mathrm{B}} \cup N_{\mathrm{C}})) \\ &\leq P(Y_{\mathrm{A}} \cup Y_{\mathrm{B}} \cup Y_{\mathrm{C}}) \leq P(Y_{\mathrm{A}}) + P(Y_{\mathrm{B}}) + P(Y_{\mathrm{C}}) < \frac{3}{4}. \end{aligned}$$

Problem 3.

(a) Suppose that $A_1, A_2, \ldots$ is a family of events such that A_i is independent of $A_{j_1} \cap \cdots \cap A_{j_\ell}$ whenever $1 \leq j_1 < \cdots < j_\ell < i$. Show that the family $A_1, A_2, \ldots$ is independent.

(b) Suppose that the family of events $A_1, \ldots, A_n$ is independent. Fix $j \in \{1, \ldots, n\}$ and let

$$B_i = \begin{cases} A_i, & i \neq j; \\ A_i^{\mathrm{c}}, & i = j. \end{cases}$$

Show that $B_1, \ldots, B_n$ is an independent family of events as well.

(c) Suppose that the family of events $A_1, \ldots, A_n$ is independent. Let $K \subseteq \{1, \ldots, n\}$ and define

$$B_i = \begin{cases} A_i, & i \notin K; \\ A_i^{\mathrm{c}}, & i \in K. \end{cases}$$

Show that $B_1, \ldots, B_n$ is an independent family of events as well.

Solution.

(a) We need to prove that for any finite subset $J \subseteq \mathbb{N}$, we have $P\left(\bigcap_{j \in J} A_j\right) = \prod_{j \in J} P(A_j)$. We will show this by induction on $|J|$. The base case $|J| = 0$ is trivial. So we can assume that the statement holds for $|J| = k$ and we need to prove it for $|J| = k + 1$. Enumerate $J = \{j_1 < \cdots < j_{k+1}\}$. Then we have

$$P\left(\bigcap_{j=1}^{k+1} A_j\right) = P\left(\left(\bigcap_{j=1}^{k} A_j\right) \cap A_{k+1}\right) = P\left(\bigcap_{j=1}^{k} A_j\right) P(A_{k+1}) = \prod_{j=1}^{k+1} P(A_j)$$

where the second identity is by the assumption in the problem and the third is by the inductive hypothesis. This completes the proof by induction.

(b) Consider some $1 \le i_1 < \cdots < i_\ell \le n$. We claim that

$$P\left(\bigcap_{k=1}^{\ell} B_{i_k}\right) = \prod_{k=1}^{\ell} P\left(B_{i_k}\right).$$

In the case that none of the i_ks is equal to j, then this identity follows immediately from the independence of the A_is. So consider the case when there is a k_* such that $i_{k_*} = j$. Then we have

$$\left(\bigcap_{k=1}^{\ell} B_{i_k}\right) \cap \left(\bigcap_{k=1}^{\ell} A_{i_k}\right) \subseteq B_{i_{k_*}} \cap A_{i_{k_*}} = A_j^{\mathrm{c}} \cap A_j = \varnothing$$

and

$$\left(\bigcap_{k=1}^{\ell} B_{i_k}\right) \cup \left(\bigcap_{k=1}^{\ell} A_{i_k}\right) = \bigcap_{\substack{k=1\\k\neq k_*}}^{\ell} A_{i_k},$$

so

$$\begin{aligned}
P\left(\bigcap_{k=1}^{\ell} B_{i_k}\right) &= P\left(\bigcap_{\substack{k=1\\k\neq k_*}}^{\ell} A_{i_k}\right) - P\left(\bigcap_{k=1}^{\ell} A_{i_k}\right)\\
&= \prod_{\substack{k=1\\k\neq k_*}}^{\ell} P(A_{i_k}) - \prod_{k=1}^{\ell} P(A_{i_k})\\
&= \prod_{\substack{k=1\\k\neq k_*}}^{\ell} P(B_{i_k}) - (1 - P(B_{i_{k_*}})) \prod_{\substack{k=1\\k\neq k_*}}^{\ell} P(B_{i_k})\\
&= \prod_{k=1}^{\ell} P(B_{i_k}),
\end{aligned}$$

which is what we needed to show.

(c) We use induction on $|K|$. The case $|K| = 0$ is obvious, so let's assume that the statement holds whenever $|K| = k$ and try to prove it for $|K| = k + 1$. Write $K = K_0 \cup \{j\}$ with $|K_0| = k$. Then the conclusion follows from part (b) where the A_is in part (b) are taken to be the B_is in this part with K replaced by K_0.

Problem 4. Exhibit events A_1, A_2, and A_3 such that $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$ but A_1, A_2, A_3 are not independent.

Full credit will be awarded for any correct example and justification. But in fact there is a very simple example, which only really relies on a particular (some would say "trivial") choice of A_1.

Proof. Take $A_1 = \varnothing$ and A_2 and A_3 to be any two events that are *not* independent. Then $P(A_1 \cap A_2 \cap A_3) = 0 = P(A_1)P(A_2)P(A_3)$, but A_1, A_2, A_3 are not independent since A_2 and A_3 are not independent. □

Problem 5. Answer Meester, exercise 1.7.24 (Simpson's paradox). Come up with another real-world situation in which you think this paradox might arise.

Solution. I find answer (2) to be more convincing. Drug I is better in both women and men. The problem is that the disease appears to be harder to cure in women than in men, and drug I was tested more in women while drug II was tested more in men than in women, so drug II was in some sense given the easier task, which makes it look better. Given a task of the same difficulty, drug

I appears to perform better. Essentially, sex was a confounding variable in the experiment which skewed the outcome. (This assumes that all other factors are equal between the trials. If there were big differences in, say, age between the different samples, this could further affect which drug is better.)

Your examples of another situation in which Simpson's paradox might apply will of course vary. One example that comes to mind is comparing GPAs between a college focused on engineering and a college focused on liberal arts. The engineering college could have more grade inflation in both engineering and liberal arts classes, but appear to have less grade inflation overall because more students there are enrolled in engineering classes which tend to have lower grades.

Problem 6 (a more precise explanation of what was going on with the islanders)**.** I have $n+1$ balls. I paint each one red with probability p and blue with probability $1-p$, independently. Then I put all of the balls in an urn and mix them up well.

(a) I draw one ball from the urn, observe that it is red, and put it back in the urn. For each $k \in \{0, \dots, n\}$, compute the probability that there are $k+1$ red balls in the urn. [*Hint*: let E_k be the event that there are $k+1$ red balls in the urn, and let A be the event that the ball you draw is red. You are trying to compute $P(E_k \mid A)$, which you can do using Bayes's rule.]

(b) Now I again draw a ball from the urn, observe that it again red, and put it back into the urn. For each $k \in \{0, \dots, n\}$, again compute the probability that there are $k+1$ red balls in the urn. Does it match the probability in part (a)? Explain why this problem is analogous to the one involving the murder on the island discussed in class (Example 1.5.14 in Meester). *If you like, you may consider only the case $n = 2$ for this part. The case of general n is extra credit (conceptually, it is the same, but you have to figure out how to compute some sums involving binomial coefficients).*

(c) Taking $p = 0.6$ and $n = 2$, perform a computer experiment to obtain numerical approximations for parts (a) and (b). Try running 1000 or 10000 trials for each part. If, when you draw a ball, the ball is not the color that we postulated observing (i.e. not red), then you should start that trial over from the beginning – this is a way to simulate the conditioning. Check that your results are close to what you computed theoretically. Attach your code and results to your submission.

Solution.

(a) We have $P(A \mid E_k) = \frac{k+1}{n+1}$ since conditional on there being $k+1$ red balls, the event A happens when one of $k+1$ red balls is drawn uniformly from among $n+1$ balls. Also, we have $P(A) = p$ since we could think of not revealing the colors of the balls until after one is drawn, at which point the drawn ball would be colored red with probability p. Finally, we have $P(E_k) = \binom{n+1}{k+1} p^{k+1}(1-p)^{n-k}$ according to the binomial distribution. Therefore, by Bayes' Rule we have

$$P(E_k \mid A) = \frac{P(A \mid E_k)P(E_k)}{P(A)} = \frac{\frac{k+1}{n+1}\binom{n+1}{k+1}p^{k+1}(1-p)^{n-k}}{p} = \binom{n}{k} p^k (1-p)^{n-k}.$$

This makes sense because the colors of the balls remaining in the urn are independent of the ones we drew, and so the probability of finding k red balls among the n remaining balls is distributed according to $\mathrm{Bin}(n, p)$.

(b) Let B be the event that the second drawn ball is red. We are now trying to compute $P(E_k \mid A \cap B)$, and we will again use Bayes' Rule. We have $P(A \cap B \mid E_k) = \left(\frac{k+1}{n+1}\right)^2$ since given that there are $k+1$ red balls, the two draws are independent. To compute $P(A \cap B)$ is a bit harder. Let's describe two ways of doing it. First, let C be the event that the two balls drawn are in fact the same ball, so we have $P(C) = \frac{1}{n+1}$. Then we have $P(A \cap B \mid C) = p$

and $P(A \cap B \mid C^c) = p^2$. Then we can compute

$$P(A \cap B) = P(A \cap B \mid C)P(C) + P(A \cap B \mid C^c)P(C^c)$$
$$= \frac{p}{n+1} + \frac{p^2 n}{n+1} = \frac{p(1+pn)}{n+1}.$$

Alternatively, we could condition on which of the E_ks occurs, and write

$$P(A \cap B) = \sum_{k=0}^{n} P(A \cap B \mid E_k)P(E_k) = \sum_{k=0}^{n} \left(\frac{k+1}{n+1}\right)^2 \cdot \binom{n+1}{k+1} p^{k+1}(1-p)^{n-k},$$

which gives the same answer after computing the sum (although this is a bit annoying to prove which is why I didn't ask you to do it). Then we can conclude by writing

$$P(E_k \mid A \cap B) = \frac{P(A \cap B \mid E_k)P(E_k)}{P(A \cap B)} = \frac{\left(\frac{k+1}{n+1}\right)^2 \cdot \binom{n+1}{k+1} p^{k+1}(1-p)^{n-k}}{\frac{p(1+pn)}{n+1}}$$
$$= \frac{\binom{n+1}{k+1}(k+1)^2 p^k (1-p)^{n-k}}{(1+pn)(n+1)} = \frac{(k+1)\binom{n}{k} p^k (1-p)^{n-k}}{1+pn}.$$

This is not the same as the answer computed in part (a). It's the same problem as the problem of John and the islanders in that conditioning on the first ball drawn is being red is like knowing that the murderer has the DNA profile, which conditions the situation on at least one of the islanders having the DNA profile, and then conditioning on the second ball being red is like conditioning on John having that same DNA profile.

(c) (code omitted from solutions)