MATH 340 – SPRING 2026 – HOMEWORK 2

Due Thursday, January 22, 2026 at 8am on Gradescope. You must justify all of your answers for full credit.

You are encouraged to collaborate with other students, but you must write up your solutions individually, without reference to notes from the collaboration. You may not search the internet or ask AI for solutions to the homework problems. Exception: it is fine to use AI to "vibe-code" the programming questions if you want, but make sure you understand what the code is doing and how to modify it.

Problem 1 (based on Meester, Exercise 1.7.14)**.** There is a single-elimination tennis tournament where each of the 2^n players is seeded at random. Assume that all of the players are evenly matched and the games are independent (and independent of the seedings as well), so each player wins each game with probability $1/2$. Given two players in the tournament, what is the probability that they play one another? Justify your answer.

Problem 2. Recall the hat game discussed in the lecture of January 13 (or see Example 1.4.5 in Meester). The purpose of this problem is to show that no deterministic strategy can win with probability greater than $3/4$. So let us fix some (unspecified) deterministic strategy. (A *deterministic* strategy is one where the players' guesses depend only on the hats that they see, not on any other pieces of information or randomness.)

(a) Let $Y_{\mathrm{A}}, Y_{\mathrm{B}}, Y_{\mathrm{C}}$ be the events that $\mathrm{A}, \mathrm{B}, \mathrm{C}$ make correct guesses, respectively. Similarly, let $N_{\mathrm{A}}, N_{\mathrm{B}}, N_{\mathrm{C}}$ be the events that $\mathrm{A}, \mathrm{B}, \mathrm{C}$ make incorrect guesses, respectively. Argue that $P(Y_{\mathrm{X}}) = P(N_{\mathrm{X}})$ for each $\mathrm{X} \in \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$.

Recall that the players win if at least one of them makes a correct guess, and none of them makes an incorrect guess, i.e. on the event $W := Y_{\mathrm{A}} \cup Y_{\mathrm{B}} \cup Y_{\mathrm{C}} \setminus (N_{\mathrm{A}} \cup N_{\mathrm{B}} \cup N_{\mathrm{C}})$.

(b) Show that if there is some $\mathrm{X} \in \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ such that $P(Y_{\mathrm{X}}) \geq 1/4$, then $P(W) \leq 3/4$.

(c) Show that if, on the other hand, there is no $X \in \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ such that $P(Y_{\mathrm{X}}) \geq 1/4$, then $P(W) \leq 3/4$ as well.

Problem 3.

(a) Suppose that $A_1, A_2, \ldots$ is a family of events such that A_i is independent of $A_{j_1} \cap \cdots \cap A_{j_\ell}$ whenever $1 \leq j_1 < \cdots < j_\ell < i$. Show that the family $A_1, A_2, \ldots$ is independent.

(b) Suppose that the family of events $A_1, \ldots, A_n$ is independent. Fix $j \in \{1, \ldots, n\}$ and let

$$B_i = \begin{cases} A_i, & i \neq j; \\ A_i^{\mathrm{c}}, & i = j. \end{cases}$$

Show that $B_1, \ldots, B_n$ is an independent family of events as well.

(c) Suppose that the family of events $A_1, \ldots, A_n$ is independent. Let $K \subseteq \{1, \ldots, n\}$ and define

$$B_i = \begin{cases} A_i, & i \notin K; \\ A_i^{\mathrm{c}}, & i \in K. \end{cases}$$

Show that $B_1, \ldots, B_n$ is an independent family of events as well.

Problem 4. Exhibit events A_1, A_2, and A_3 such that $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$ but A_1, A_2, A_3 are not independent.

Full credit will be awarded for any correct example and justification. But in fact there is a very simple example, which only really relies on a particular (some would say "trivial") choice of A_1.

Problem 5. Answer Meester, exercise 1.7.24 (Simpson's paradox). Come up with another real-world situation in which you think this paradox might arise.

Problem 6 (a more precise explanation of what was going on with the islanders)**.** I have $n + 1$ balls. I paint each one red with probability p and blue with probability $1 - p$, independently. Then I put all of the balls in an urn and mix them up well.

(a) I draw one ball from the urn, observe that it is red, and put it back in the urn. For each $k \in \{0, \ldots, n\}$, compute the probability that there are $k + 1$ red balls in the urn. [*Hint*: let E_k be the event that there are $k + 1$ red balls in the urn, and let A be the event that the ball you draw is red. You are trying to compute $P(E_k \mid A)$, which you can do using Bayes's rule.]

(b) Now I again draw a ball from the urn, observe that it again red, and put it back into the urn. For each $k \in \{0, \ldots, n\}$, again compute the probability that there are $k + 1$ red balls in the urn. Does it match the probability in part (a)? Explain why this problem is analogous to the one involving the murder on the island discussed in class (Example 1.5.14 in Meester). *If you like, you may consider only the case $n = 2$ for this part. The case of general n is extra credit (conceptually, it is the same, but you have to figure out how to compute some sums involving binomial coefficients).*

(c) Taking $p = 0.6$ and $n = 2$, perform a computer experiment to obtain numerical approximations for parts (a) and (b). Try running 1000 or 10000 trials for each part. If, when you draw a ball, the ball is not the color that we postulated observing (i.e. not red), then you should start that trial over from the beginning – this is a way to simulate the conditioning. Check that your results are close to what you computed theoretically. Attach your code and results to your submission.