# MATH 340 – SPRING 2026 – HOMEWORK 3

Due Thursday, January 29, 2026 at 8am on Gradescope. You must justify all of your answers for full credit.

You are encouraged to collaborate with other students, but you must write up your solutions individually, without reference to notes from the collaboration. You may not search the internet or ask AI for solutions to the homework problems. Exception: it is fine to use AI to "vibe-code" the programming questions if you want, but make sure you understand what the code is doing and how to modify it.

**Problem 1.**

(a) Suppose that $\Omega = \Omega_1 \times \cdots \times \Omega_n$ and $p(\omega_1, \ldots, \omega_n) = p_1(\omega_1) \cdots p_n(\omega_n)$. Let $X_i(\omega) = f_i(\omega_i)$ for some function $f_i$. Prove that $X_1, \ldots, X_n$ are independent random variables.

(b) Let $X_1, X_2, X_3$ be independent random variables, and let $g\colon \mathbb{R}^2 \to \mathbb{R}$ and $h\colon \mathbb{R} \to \mathbb{R}$ be functions. Show that $g(X_1, X_2)$ and $h(X_3)$ are independent. State and prove a generalization of this result.

**Problem 2.** Let $\Omega = \{0, 1, 2, 3\}$ be a sample space equipped with probability measure given by $p(0) = 1/2$, $p(1) = 1/6$, $p(2) = 1/6$, $p(3) = 1/6$.

(a) Exhibit two random variables $X$ and $Y$ defined with respect to the sample space $\Omega$ such that $X \sim \mathrm{Ber}(1/3)$ and $Y \sim \mathrm{Ber}(1/3)$ but $X$ and $Y$ are not the same random variable.

(b) Show that there is no random variable $X$ defined on $\Omega$ such that $X \sim \mathrm{Bin}(2, 1/2)$.

(c) Show that if three random variables $X, Y, Z$ on $\Omega$ are independent, then there must be some $x \in \mathbb{R}$ such that either $P(X = x) = 1$, $P(Y = x) = 1$, or $P(Z = x) = 1$.

**Problem 3.** Recall the airplane problem from Homework 1, Problem 3. Let $X_n$ be the number of passengers who sit in their assigned seat. For each $k \geq 0$, compute $\lim_{n\to\infty} p_{X_n}(k)$. [*Hint*: if exactly $k$ people sit in their assigned seats, then none of the remaining $n-k$ people can be sitting in their assigned seats, so you can use the result of Homework 1, Problem 3 with $n$ replaced by $n-k$.]

**Problem 4.** Suppose that $X \sim \mathrm{Bin}(n, p)$ and $Y \sim \mathrm{Bin}(m, p)$ are independent. Compute (with proof) the pmf of the random variable $X + Y$.

**Problem 5** (some problems about the geometric distribution)**.**

(a) Let $X \sim \mathrm{Geom}(p)$. Prove that $P(X = m + k \mid X > m) = P(X = k)$. Explain why this is called the *memoryless property* of the geometric distribution.

(b) Let $X_1, \ldots, X_r \sim \mathrm{Geom}(p)$ be independent. Show that $X_1 + \cdots + X_r$ has a negative binomial distribution with parameters $(p, r)$.

**Problem 6.** Let $s > 1$. Let $X$ be a random variable such that $p_X(k) = k^{-s}/\zeta(s)$ for $k = 1, 2, 3, \ldots$ and $p_X(x) = 0$ for other $x \in \mathbb{R}$. Here the normalizing factor is

$$\zeta(s) := \sum_{k=1}^{\infty} k^{-s}.$$

This function is also known as the *Riemann zeta function* and it is very important in number theory. For each prime number $q$, let $Y_q$ be the random variable such that the prime factorization of $X$ (which is unique) can be written as

$$X = \prod_{q \text{ prime}} q^{Y_q}.$$

(a) Show that $Y_q + 1 \sim \mathrm{Geom}(p_q)$ and compute $p_q$.
(b) Show that the family $\{Y_q\}_{q \text{ prime}}$ is an independent family of random variables. [*Hint*: it suffices to show that for any distinct prime numbers $q_1, \ldots, q_n$ and any nonnegative integers $m_1, \ldots, m_n$, the events $\{Y_{q_1} \geq m_1\}, \ldots, \{Y_{q_n} \geq m_n\}$ are independent.]
(c) By using the last two parts, along with a limit lemma proved in class, show that

$$\frac{1}{\zeta(s)} = P(X = 1) = \prod_{q \text{ prime}} (1 - q^{-s}).$$

This is the famous *Euler product formula* for the Riemann zeta function.