MATH 340 – SPRING 2026 – HOMEWORK 4

Due Thursday, February 5, 2026 at 8am on Gradescope. You must justify all of your answers for full credit.

You are encouraged to collaborate with other students, but you must write up your solutions individually, without reference to notes from the collaboration. You may not search the internet or ask AI for solutions to the homework problems.

Problem 1. In a certain community (which I make no claim actually exists, or corresponds to any real community), families desire to have as few children as possible, but they want to have at least one female child. Let us assume that there are no multiple births, and that each child is assigned either male or female at birth, with equal probability, and that the sexes assigned to all births are independent. Each family has children until a girl is born, and then stops.

(a) What is the expected number of male and female children in each family? How does this reproduction strategy affect the balance of males to females in the community?
(b) Conditional on the total number of children in a family being n, what is the conditional distribution of the number of boys in the family?

In a different community, after having each child, families decide to keep having children with probability $p \in (0, 1)$, independent of everything else (i.e. they flip an unfair coin after each birth).

(c) Conditional on the total number of children in a family being n, what is the conditional distribution of the number of boys in the family?
(d) What is the expected number of male and female children in each family? How does this reproduction strategy affect the balance of males to females in the community? [*Hint*: if N is the total number of children and M is the number of boys, start by computing $E[M \mid N = n]$.]

Let's now generalize the previous examples. Suppose in another community, there is some reproduction strategy that we do not know. All we know is that this community does not practice sex-selective abortion, so after having each child, parents decide whether or not to (try to) have another child without knowing what the gender of that child would be. We can model this situation as follows. Let N be the total number of children that the family has. Assume that $N \leq 20$ almost surely. Let ξ_i be the indicator of the event that the ith child would be female if that child were actually born, so $\xi_1, \dots, \xi_{20}$ are independent.

(e) Explain why, based on the description given, we should make the modeling assumption that, for each i, the random variables $\mathbf{1}\{N \geq i\}, \xi_i, \xi_{i+1}, \dots, \xi_{20}$ are independent. Also, explain why the total number of girls that the family has is

$$G := \sum_{i=1}^{20} \mathbf{1}\{N \geq i\}\xi_i$$

and the total number of boys that the family has is

$$B := \sum_{i=1}^{20} \mathbf{1}\{N \geq i\}(1 - \xi_i).$$

(f) Compute $E[G - B]$. How does the strategy affect the balance of males to females in the community?

Problem 2 (Airplane problem again)**.** Recall again the chaotic airplane passengers of Homework 1, Problem 3. Compute the expected number of people who sit in their assigned seats. [*Hint*: use linearity of expectation and don't work too hard.]

Problem 3. Let $\xi_1, \dots, \xi_n$ be i.i.d. Rademacher random variables. Compute

$$R_n := E\left[\left(\sum_{i=1}^{n} \xi_i\right)^4\right].$$

Find an α such that R_n/n^α approaches a finite limit as $n \to \infty$. [*Hint*: expand the sum as $(\sum_{i=1}^n \xi_i)^4 = \sum_{i_1,i_2,i_3,i_4=1}^n \xi_{i_1}\xi_{i_2}\xi_{i_3}\xi_{i_4}$, and then separate the right hand side into different parts depending on which of the i_js are the same or different.]

Problem 4 (second moment method)**.** Suppose that X is a random variable such that $X \geq 0$ almost surely. Show that

$$P(X = 0) \leq \frac{\mathrm{Var}(X)}{(E[X])^2}.$$

[*Hint*: start with the inequality $P(X = 0) \leq P(|X - E[X]| \geq E[X])$.]

Problem 5. A *graph* consists of a set $\mathcal{V}$ of *vertices* and a set of *edges* $\mathcal{E} \subseteq \binom{V}{2}$ (so an edge is an ordered pair of distinct vertices). We think of the edge set as the set of vertices that are "adjacent" to each other in the graph. For $n \in \mathbb{N}$, let $\mathcal{G}$ be a *random* graph with vertex set $\mathcal{V} = \{1, \dots, n\}$ and each edge present independently with probability p. For $i \neq j \in \mathcal{V}$, let $\xi_{i,j}$ denote the indicator of the event that $\{i, j\} \in E$, i.e. that i and j are adjacent. This means that the family $(\xi_{i,j})_{\{i,j\} \in \binom{\{1,\dots,n\}}{2}}$ is a family of $\binom{n}{2}$ independent $\mathrm{Ber}(p)$ random variables.

(a) Compute $E|\mathcal{E}|$, the expected number of edges in the graph, in terms of n and p.

(b) A *triangle* in a graph is a set of distinct vertices $i, j, k \in \mathcal{V}$ such that $\{i, j\}, \{j, k\}, \{i, k\} \in \mathcal{V}$ (i.e. all three vertices are connected). Let T be the number of triangles in $\mathcal{G}$. Compute $E[T]$. [*Hint*: for i, j, k, let $\eta_{i,j,k}$ be the indicator of the event that i, j, k form a triangle, and start by computing $E[\eta_{i,j,k}]$.]

(c) Suppose that $p = p_n$ is such that $np_n \to 0$ as $n \to \infty$. Show that the probability that $\mathcal{G}$ contains at least one triangle goes to 0 as $n \to \infty$. [*Hint*: previous part and Markov's inequality.]

(d) Compute $\mathrm{Var}(T)$. [*Hint*: this is somewhat similar to Problem 3. The $\eta_{i,j,k}$s are not all independent, but many of them are.]

(e) Suppose that $p = p_n$ is such that $np_n \to \infty$ as $n \to \infty$. Show that the probability that $\mathcal{G}$ contains at least one triangle goes to 1 as $n \to \infty$. [*Hint*: use Problems 4, 5(b), and 5(d).]