office hours after class today
& 3:30 tomorrow.

please send me email to introduce yourself w/ background, goals in the class, etc.

UNION BOUND. (Boole's inequality)

If $E_1, E_2, \ldots, E_n$ are events over a probability space $(\Omega, \mathbb{P})$, then

$$\mathbb{P}(E_1 \cup E_2 \cup \cdots \cup E_n) \leq \sum_{i=1}^{n} \mathbb{P}(E_i) \qquad (*)$$

Recall: if the E_i's are disjoint, then equality holds.

Proof Induction.

Base case ($n=1$): $\mathbb{P}(E_1) = \mathbb{P}(E_1)$. ✓

Inductive step: assume that $(*)$ holds for $n=k$. Let's try to prove it for $n=k+1$.

$$\begin{aligned}
&\mathbb{P}(E_1 \cup \cdots \cup E_{n+1}) \\
&= \mathbb{P}((E_1 \cup \cdots \cup E_n) \cup E_{n+1}) \\
&= \mathbb{P}(E_1 \cup \cdots \cup E_n) + \mathbb{P}(E_{n+1}) \\
&\qquad - \mathbb{P}((E_1 \cup \cdots \cup E_n) \cap E_{n+1}) \\
&\leq \sum_{i=1}^{n} \mathbb{P}(E_i) + \mathbb{P}(E_{n+1}) \\
&= \sum_{i=1}^{n+1} \mathbb{P}(E_i).
\end{aligned}$$

$\square$

Conditional probability

Defn. Suppose that A and B are events and $P(B)>0$. Then the conditional probability of A given B is

$$P(A\mid B)=\frac{P(A\cap B)}{P(B)}.$$

\mid

Remark. If (Ω,P) is a probability space, and $B\subseteq\Omega$ has $P(B)>0$, then $(B, P(\cdot\mid B))$ is another probability space.

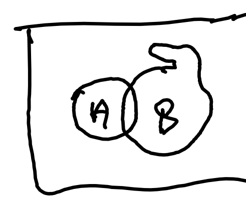
$$p(\omega\mid B)=\begin{cases}\dfrac{p(\omega)}{P(B)} & \text{if } \omega\in B\\ 0 & \text{otherwise.}\end{cases}$$

$$=P(\{\omega\}\mid B)$$

Proof of Remark

- $p(\omega\mid B)\geq 0$ is clear.
- $p(\omega\mid B)\leq 1$ because either $\omega\notin B$ so it's 0 or $\omega\in B$ and
  $$P(B)=\sum_{\tilde\omega\in B} p(\tilde\omega)\geq p(\omega).$$
- $$\sum_{\omega\in B} p(\omega\mid B)=\sum_{\omega\in B}\frac{p(\omega)}{P(B)}=1.$$

$\square$



Ω

Example. Flip a coin twice.

Let A be the event that that the first flip is tails and let B be the event that at least one of the flips is tails.

$\Omega = \{HH, HT, TH, TT\}$.

$p(\omega) = \frac{1}{4}$ for each $\omega \in \Omega$.

$A = \{TH, TT\}$

$B = \{HT, TH, TT\}$.

$P(A) = \frac{1}{2}$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{2}{3}.$$

Example ① Your friend has two cats. One of them is male. What is the probability that they are both male?

② Your friend has two cats. The older one is male. What is the prob. that they are both male?

① 1/3. $P(\text{both male} \mid \text{one of them is male})$

$$= \frac{P(MM)}{P(MM, MF, FM)} = \frac{1}{3}.$$

② 1/2. $P(\text{both male} \mid \text{older one is male})$

$$= \frac{P(MM)}{P(MM, MF)} = \frac{1}{2}.$$

<u>Example</u>. Alice, Bob, Carol are in prison. The warden will randomly give each of them a hat, either red or blue. w/ equal prob. The players must each write on a slip of paper either a color, or "PASS".

The players are all released if

1. at least one of them guesses their own hat color correctly

and 2. nobody guesses incorrectly.

What is the best strategy? Can they do better than 50-50?

<u>50-50</u>: A guesses red, others pass.

you might think you can't do better, because any player who guesses can't guess correctly w.p. $> 1/2$.

<u>Another strategy</u>: each player looks at the other two hats. if they are the same color, they guess the other color. Otherwise, they pass.

$\Omega = \{RRR, RRB, RBR, BRR,$
$BBR, BRB, RBB, BBB\}$

in all outcomes except for RRR, BBB, they succeed. so $P(\text{success}) = \frac{3}{4}$.

$P(A \text{ is red} \mid B \text{ and } C \text{ same color}) = \frac{1}{2}$

$P(A \text{ different from } B \& C \mid B \text{ and } C \text{ same}) = \frac{1}{2}$.

failure events overlap so overall prob. of failure is less.

Conditional prob. is very useful for computing prob:

We say that a countable sequence of events $B_1, B_2, \ldots$ is a <u>partition</u> of Ω if the B_is are pairwise disjoint and $\bigcup_{i=1}^{\infty} B_i = \Omega$.

<u>Theorem.</u> Let $B_1, B_2, \ldots$ be a partition of Ω such that $P(B_i) > 0$. Let A be any event. Then

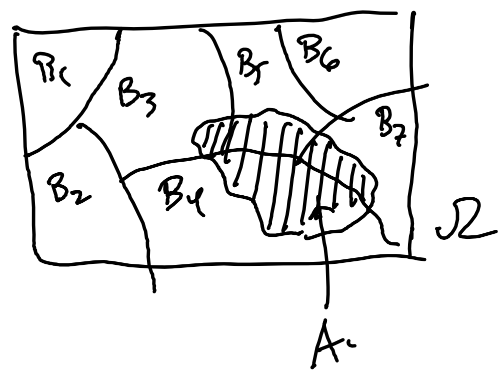
$$P(A) = \sum_i P(A \mid B_i) P(B_i)$$

<u>Proof.</u> $A = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup \cdots$

& the events on the rhs are all disjoint. So:

$$P(A) = \sum_{i=1}^{\infty} P(A \cap B_i)$$ → because B_is are disjoint

$$= \sum_{i=1}^{\infty} P(A \mid B_i) P(B_i). \quad \square$$



Prop Let $A_1, A_2, \dots, A_n$ be events, such that $P(A_1 \cap \dots \cap A_{n-1}) > 0$.

Then

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)P(A_2|A_1)P(A_3|A_1\cap A_2) \cdots P(A_n|A_1\cap\cdots\cap A_{n-1}).$$

Proof. Induction. Base case $n=1$. clear.

Inductive step:

$$P(A_1\cap \dots \cap A_{n+1}) = P(A_{n+1}|A_1\cap\dots\cap A_n) \cdot P(A_1\cap \dots \cap A_n)$$

$$= P(A_{n+1}|A_1\cap\dots\cap A_n) \cdot P(A_n|A_1\cap\dots\cap A_{n-1}) \cdot \dots \cdot P(A_2|A_1)P(A_1).$$

inductive hypothesis

Example. A population of people has 1% of people with some disease. There is a test for the disease. that is 90% sensitive

i.e. $P(\text{test positive} \mid \text{sick}) = 0.9$

and 80% specific

i.e. $P(\text{test negative} \mid \text{healthy}) = 0.8$.

Suppose we pick a random person and they test positive? What is the prob that they are sick?

$$P(\text{sick} \mid \text{test positive}) = \frac{P(\text{sick \& test positive})}{P(\text{test positive})}$$

$$= \frac{P(\text{test positive} \mid \text{sick})\, P(\text{sick})}{P(\text{test positive} \mid \text{sick})\, P(\text{sick}) + P(\text{test positive} \mid \text{healthy})\, P(\text{healthy})}$$

$$= \frac{0.9 \cdot 0.01}{0.9 \cdot 0.01 + 0.2 \cdot 0.99} = 4.76\%.$$

Special case of Bayes' Rule

Let $B_1, B_2, \ldots, B_n$ be a partition of Ω s.t. $P(B_i) > 0$ for all i. Let A be any event w/ $P(A) > 0$. Then

$$P(B_i \mid A) = \frac{P(A \mid B_i) P(B_i)}{\sum_{j=1}^{n} P(A \mid B_j) P(B_j)}.$$

Proof. $P(B_i \mid A) = \dfrac{P(B_i \cap A)}{P(A)}$

$$= \frac{P(A \mid B_i) P(B_i)}{\sum_{j=1}^{n} P(A \mid B_j) P(B_j)}. \quad \square.$$

($n = \infty$ also OK).