Independence. Two events A and B are called independent if $P(A \cap B) = P(A)P(B)$.

[if $P(B) > 0$, this is equivalent to $P(A|B) = P(A)$.]

More generally, a family of events $A_1, A_2, \ldots$ is called independent if for any finite collection of indices J, we have

$$P\left(\bigcap_{j \in J} A_j\right) = \prod_{j \in J} P(A_j).$$

Prop. Suppose $P(B) \in \{0, 1\}$. Then B is independent of A for any event A.

Proof. Say $P(B) = 0$. Then

$$P(A \cap B) \le P(B) = 0$$

so $P(A \cap B) = 0$.

And $P(A)P(B) = 0$ as well.

In case $P(B) = 1$, you can try that yourself.

Example: Let's consider flipping two coins.

$\Omega = \{HH, HT, TH, TT\}$.

$A_1 = \{\text{first coin is heads}\} = \{HH, HT\}$

$A_2 = \{\text{second coin is heads}\} = \{HH, TH\}$

$B = \{\text{exactly one heads}\} = \{HT, TH\}$.

Now:
- A_1 is independent of A_2.
- A_1 is independent of B.
- A_2 is independent of B.

So these events are "pairwise independent", i.e. any pair is independent.

But is the family A_1, A_2, B independent?

No! $P(A_1 \cap A_2 \cap B) = 0$.

$\therefore$ Pairwise independence does NOT imply independence

Question: suppose A and B are independent, and $P(C) > 0$. Then is it true that

$P(A \cap B | C) = P(A|C) P(B|C)$.

No. $P(A_1|B) = \frac{1}{2}$, $P(A_2|B) = \frac{1}{2}$

$P(A_1 \cap A_2 | B) = 0$.

Prop. If $\Omega = \Omega_1 \times \cdots \times \Omega_n$ is equipped with pmf

$$p(\omega_1, \dots, \omega_n) = p_1(\omega_1) \cdots p_n(\omega_n),$$

then whenever $A_i \subseteq \Omega_i$, if we define $B_i = \Omega_1 \times \cdots \times \Omega_{i-1} \times A_i \times \Omega_{i+1} \times \cdots \times \Omega_n$

then $B_1, \dots, B_n$ is an independent family of events.

Proof. Let $J \subseteq \{1, \dots, n\}$

$$\bigcap_{j \in J} B_j = \bigtimes_{j=1}^{n} C_j.$$

$$C_j = \begin{cases} A_j & \text{if } j \in J. \\ \Omega_j & \text{o/w}. \end{cases}$$

so $$P\Big(\bigcap_{j \in J} B_j\Big) = \sum_{\omega \in \bigcap_{j \in J} B_j} p(\omega)$$

$$= \sum_{\omega_1 \in C_1} \cdots \sum_{\omega_n \in C_n} p(\omega_1, \dots, \omega_n)$$

$$= \sum_{\omega_1 \in C_1} \cdots \sum_{\omega_n \in C_n} p_1(\omega_1) \cdots p_n(\omega_n)$$

$$= \Big(\underbrace{\sum_{\omega_1 \in C_1} p_1(\omega_1)}_{= P_1(C_1)}\Big) \cdots \Big(\sum_{\omega_n \in C_n} p_n(\omega_n)\Big)$$

$$= \overbrace{P(C_1 \times \Omega_2 \times \cdots \times \Omega_n)} \cdots$$

$$P(\Omega_1 \times \cdots \times \Omega_{n-1} \times C_n)$$

$$= \prod_{j \in J} P(B_j)$$

Example Suppose we want to flip n unfair coins "independently". each coin has probability p_i of being heads. We can take the sample space to be

$$\Omega = \{H,T\}^n = \underbrace{\{H,T\} \times \cdots \times \{H,T\}}_{n \text{ times}}.$$

$$p(\omega_1, \ldots, \omega_n) = \prod_{i=1}^{n} \left(p_i \mathbb{1}\{\omega_i = H\} + (1-p_i)\mathbb{1}\{\omega_i = T\} \right)$$

$$\mathbb{1}\{A\} = \begin{cases} 1 & \text{if } A \text{ is true/occurs} \\ 0 & \text{o/w.} \end{cases}$$

Then Prop $\Rightarrow$ the events that the ith flip is heads are all independent.

in the language of the prop, could take $A_i = \{H\} \subseteq \{H,T\}$.

$B_i = \{i\text{th flip is heads}\}$

$= \Omega_1 \times \cdots \times \Omega_{i-1} \times A_i \times \Omega_{i+1} \times \cdots \times \Omega_n$.

Example. What is the probability of the event A_k that the first heads occurs on the kth flip?

$$A_k = B_1^c \cap B_2^c \cap \cdots \cap B_{k-1}^c \cap B_k.$$

so $P(A_k) = P(B_1^c) P(B_2^c) \cdots P(B_{k-1}^c) P(B_k)$

$= (1-p_1)(1-p_2) \cdots (1-p_{k-1}) \cdot p_k$

Need another prop:

if $A_1, \ldots, A_n$ are independent, then so are $A_1, \ldots, A_n, A_1^c, \ldots, A_n^c$.

this will also be on the HW.

Extended example. An island has $n+2$ inhabitants. One of them is murdered. So there are $n+1$ suspects. Police from the mainland arrive, and find some DNA evidence at the scene of the crime. Scientists say that that this DNA is consistent w/ the DNA of p fraction of all people.

The police find a random islander John. They test him and find that his DNA is a match. What is the probability that John is guilty?

<u>Two approaches</u>: first, let's figure out how many people on the island have the same DNA profile as John.

Let $A_k = \{$exactly k other people w/ this DNA$\}$.

$$P(A_k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

$P(\text{J. is guilty}) = \quad \ldots$

$$P(\text{John is guilty}) = \sum_{k=0}^{n} P(\text{guilty} \mid A_k) P(A_k)$$

$P(A_k)$ is actually not $\binom{n}{k} p^k (1-p)^{n+1}$

$$= \sum_{k=0}^{n} \frac{1}{k+1} \cdot \binom{n}{k} p^k (1-p)^{n-k}.$$

$$= \frac{1}{p(n+1)} \sum_{k=0}^{n} \frac{(n+1)!}{(k+1)!(n-k)!} p^{k+1} (1-p)^{n+1-(k+1)}$$

$$= \frac{1}{p(n+1)} \sum_{k=0}^{n} \binom{n+1}{k+1} p^{k+1} (1-p)^{(n+1)-(k+1)}$$

$$= \frac{1}{p(n+1)} \sum_{k=1}^{n+1} \binom{n+1}{k} p^k (1-p)^{n+1-k}$$

$$(p + (1-p))^{n+1} = 1.$$

$$= \frac{1}{p(n+1)} \left[\sum_{k=0}^{n+1} \binom{n+1}{k} p^k (1-p)^{n+1-k} - (1-p)^{n+1} \right].$$

$$= \frac{1}{p(n+1)} \left[1 - (1-p)^{n+1} \right].$$

This concludes approach #1.

Approach 2: Bayes' rule.

$G = \{\text{John is guilty}\}$

$E = \{\text{finding DNA like John's at the crime scene}\}.$

before we saw the DNA,

$P(G) = \frac{1}{n+1}$. goal: find $P(G|E)$.

$P(E|G^c) = p \qquad P(E|G) = 1.$

$$P(G|E) = \frac{P(E|G)P(G)}{P(E|G)P(G) + P(E|G^c)P(G^c)}$$

$$= \frac{1/(n+1)}{1/(n+1) + p \cdot \frac{n}{n+1}} = \frac{1}{1+pn}.$$