DNA problem: The extra conditioning is coming from the fact that we already knew that someone on the island had the DNA profile.

Today: random variable.

Informally: a random variable represents some (real-valued) quantity that we think of as random.

e.g.
- tomorrow's temperature
- # of heads when we flip 3 coins
- the number of times we have to roll a die before rolling a 6

...

Formally:

Defn. A random variable on a probability space (Ω, P) is a (real-valued) function on Ω.

Example: flip a coin n times. we bet \$1 on each flip being H.

Sample space $\Omega = \{H,T\}^n$.

winnings $S(\omega) = \sum_{i=1}^{n} \left(\mathbb{1}\{\omega_i = H\} - \mathbb{1}\{\omega_i = T\} \right)$

<u>Note</u>: could also have taken $\Omega = \{\pm 1\}^n$, 1 for heads, -1 for tails, and then defined $S(\omega) = \sum_{i=1}^{n} \omega_i$.

So in the example

S is a r.v.

$$P(S=-n) = 2^{-n}.$$

$$P(\underbrace{S=n-2k}_{P(k \text{ tails})}) = 2^{-n}\binom{n}{k}.$$

<u>Notation</u>. We are often interested in the event that a random variable takes a certain value, or lies in a certain set. So we define for all $x \in \mathbb{R}$

$$\{X = x\} := \{\omega \in \Omega : X(\omega) = x\}.$$

$\subseteq \Omega$. (is an event).

Similarly $P(X=x) = P(\{X=x\})$

$= P(\{\omega : X(\omega) = x\})$.

Given a random variable X, the probability mass function (pmf) of X is

$$p_X(x) = P(X=x).$$

$\{X(\omega) : \omega \in \Omega\}$

<u>Note</u>: $(X(\Omega), p_X)$ defines another probability space. [Exercise: check]

[So: a r.v. is a map of probability spaces]

In the example, I could have taken

$$\tilde{\Omega} = \{-n, -n+2, \dots, n-2, n\}$$

$$p(n-2k) = \binom{n}{k} 2^{-n}.$$

Defn. The cumulative distribution function (cdf) of X is the function

$$F_X : \mathbb{R} \to [0,1] \text{ given by}$$

$$F_X(x) = \mathbb{P}(X \le x) = \sum_{y \le x} p_X(y).$$

Prop. Two r.v.s have the same pmf if and only if they have the same cdf. In this case, we say that they have the same distribution.

Example. $\Omega = \{H,T\}^2$. P = uniform

$$X_1(\omega) = X_1(\omega_1,\omega_2) = \mathbb{1}\{\omega_1 = H\}$$

$$X_2(\omega) = \mathbb{1}\{\omega_2 = T\}.$$

$$P(\{(\omega_1,\omega_2) : \omega_1 = T\}) = P(\{(T,H),(T,T)\}).$$

$$p_{X_1}(0) = p_{X_1}(1) = \tfrac{1}{2}.$$

$$p_{X_2}(0) = p_{X_2}(1) = \tfrac{1}{2}$$

So X_1 & X_2 have the same distribution. But they are different r.v.s.

Proof of prop. ($\Rightarrow$) Suppose X and Y are such that $p_X(x) = p_Y(x)$ for all $x \in \mathbb{R}$. Then

$$F_X(x) = P(X \le x) = \sum_{y \le x} p_X(y)$$

$$= \sum_{y \le x} p_Y(y) = P(Y \le x)$$

$$= F_Y(x).$$

($\Leftarrow$) proof later.

We give some common distns names.

"$X \sim \mu$" means that the distn of X is μ

Example We say that

$X \sim Ber(p)$ "Bernoulli distribution"

if $p_X(0) = 1-p$

$p_X(1) = p.$

e.g. if $\Omega = \{H, T\}$

$p(H) = p$, $p(T) = 1-p$

$X(\omega) = \mathbb{1}\{\omega = H\}$

then $X \sim Ber(p)$.

Also: $X \sim$ Rademacher

if $p_X(-1) = 1/2$

$p_X(1) = 1/2$.

So if $X \sim Ber(\frac{1}{2})$ then $2X - 1 \sim$ Rademacher *proof.*

if $X \sim Ber(\frac{1}{2})$

then $P(2X-1=-1)$

$= P(X=0) = \frac{1}{2}$

$P(2X-1=1)$

$= P(X=1) = \frac{1}{2}.$

if we flip n unfair coins w.p. p of each being heads, independently, then if X is the number of heads, then

$X \sim Bin(n,p)$.

Another example

$X \sim Bin(n,p)$ "binomial"

if $$p_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k}, & k \in \{0,\dots,n\} \\ 0, & o/w. \end{cases}$$

Suppose $X_n \sim Bin(n, \lambda/n)$ for some fixed $\lambda > 0$.

Let $p_n = p_{X_n}$. Then for $x \in \{0,\dots,n\}$,

$$p_n(x) = \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}.$$

$$= \frac{1}{x!} \left(1-\frac{\lambda}{n}\right)^n \left(\frac{\lambda}{1-\lambda/n}\right)^x \cdot \frac{n(n-1)\cdots(n-x+1)}{n^x}.$$

$$\xrightarrow[n\to\infty]{} \frac{1}{x!} e^{-\lambda} \lambda^x$$

we say $X \sim \text{Poisson}(\lambda)$

if $p_X(x) = \begin{cases} e^{-\lambda}\dfrac{\lambda^x}{x!} & x \in \mathbb{N} \\ 0 & o/w. \end{cases}$

if $\lambda = 1$ then

$$p_X(x) = \frac{e^{-1}}{x!}$$

in particular $P(X=0) = e^{-1}$.

In the airplane problem

n people, each one has prob. $\frac{1}{n}$

of sitting in the correct seat.

The people on the airplane are <u>not</u> independent. but they <u>almost</u> are!

we say that $X \sim \text{Geom}(p)$

if $p_X(x) = \begin{cases} p(1-p)^{x-1} & x \in \mathbb{N} \\ 0 & o/w. \end{cases}$

This is the probability of the first success in a sequence of indep. $\text{Ber}(p)$ trials happening on the xth trial