Central Limit Theorem (very special case)

Let $X_1, X_2, \dots, X_n \overset{iid}{\sim}$ Rademacher, $S_n = X_1 + \dots + X_n$ is the sum. Then for any $a < b$, we have

$$\lim_{n\to\infty} P\left(a \le \frac{S_n}{\sqrt{n}} \le b\right) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx.$$

In fact: it is not necessary for X_is to be Rademacher r.v.s, in fact any distribution is fine as long as $\mathrm{Var}(X_i) = 1$. We will prove this later. [UNIVERSALITY]

Lemma (Stirling's formula.)

For all $n \geq 1$, we have

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \le n! \le 2\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Moreover,

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1.$$

Now we can begin the proof (sketch) of CLT.

Notation

$a_n \sim b_n$ means that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

So Sterling says that $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

Let's first consider S_{2n}.

$X_i \sim$ Rademacher

$Y_i = \frac{X_i+1}{2} \sim Ber\left(\frac{1}{2}\right)$.

So $T_{2n} = \frac{S_{2n}+2n}{2} = \sum_{i=1}^{2n} Y_i \sim Bin\left(2n, \frac{1}{2}\right)$

$$\text{So } P(S_{2n} = 2k) = P(2T_{2n} - 2n = 2k)$$
$$= P(T_{2n} = n+k)$$
$$= \binom{2n}{n+k} 2^{-2n}$$
$$= \frac{(2n)!\, 2^{-2n}}{(n+k)!(n-k)!} \sim \frac{\cancel{2^{-2n}}\sqrt{2\pi(2n)}\left(\frac{2n}{\cancel{e}}\right)^{2n}}{\sqrt{2\pi(n+k)}\left(\frac{n+k}{\cancel{e}}\right)^{n+k} \cdot \sqrt{2\pi(n-k)}\left(\frac{n-k}{\cancel{e}}\right)^{n-k}}$$

$$= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2n}{(n+k)(n-k)}} \cdot \frac{n^{2n}}{(n+k)^{n+k}(n-k)^{n-k}}$$

$$= \frac{1}{\sqrt{\pi n}}\sqrt{\frac{n^2}{(n+k)(n-k)}} \cdot \left(\frac{n}{n+k}\right)^{n+k}\left(\frac{n}{n-k}\right)^{n-k}$$

Remember that we are interested in the regime $2k \approx x\sqrt{2n}$.
So define $x = \frac{2k}{\sqrt{2n}}$, so $k = x\sqrt{\frac{n}{2}}$

$$= \frac{1}{\sqrt{\pi n}}\underbrace{\sqrt{\frac{n^2}{(n+x\sqrt{n/2})(n-x\sqrt{n/2})}}}_{\to 1} \cdot \left(\frac{n}{n+x\sqrt{n/2}}\right)^{n+x\sqrt{n/2}}\left(\frac{n}{n-x\sqrt{n/2}}\right)^{n-x\sqrt{n/2}}$$

$$\sim \frac{1}{\sqrt{\pi n}}\left(\frac{n}{n+x\sqrt{n/2}}\right)^{n+x\sqrt{n/2}}\left(\frac{n}{n-x\sqrt{n/2}}\right)^{n-x\sqrt{n/2}}$$

$$= \frac{1}{\sqrt{\pi n}}\left(\frac{1}{1+x\sqrt{\frac{1}{2n}}}\right)^{n+x\sqrt{n/2}}\left(\frac{1}{1-x\sqrt{\frac{1}{2n}}}\right)^{n-x\sqrt{n/2}}$$

$$= \frac{1}{\sqrt{\pi n}}\underbrace{\left(\frac{1}{1-\frac{x^2}{2n}}\right)^{n}}_{\to e^{x^2/2}} \cdot \underbrace{\left(1+\frac{x}{\sqrt{2n}}\right)^{-x\sqrt{n/2}}}_{e^{-x^2/2}}\underbrace{\left(1-\frac{x}{\sqrt{2n}}\right)^{x\sqrt{n/2}}}_{e^{-x^2/2}}$$

$$\sim \frac{1}{\sqrt{\pi n}} e^{-x^2/2}$$

Now need to compute

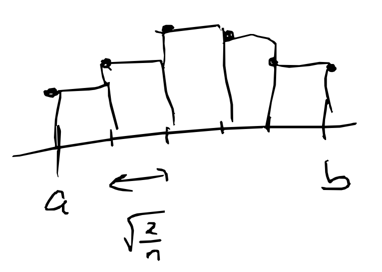
$$\lim_{n\to\infty}\left(1+\frac{x}{\sqrt{2n}}\right)^{-x\sqrt{n/2}}$$
$$= \lim_{n\to\infty}\left(1+\frac{x^2/2}{x\sqrt{n/2}}\right)^{-x\sqrt{n/2}}$$
$$= e^{-x^2/2}$$

Summary: $P(S_{2n} = x\sqrt{2n}) \sim \frac{1}{\sqrt{\pi n}} e^{-x^2/2}$.

So: $P(a\sqrt{2n} \le S_{2n} \le b\sqrt{2n})$

$$= \sum_{k:\, a\sqrt{2n} \le 2k \le b\sqrt{2n}} P\Big(S_{2n} = \underbrace{\tfrac{2k}{\sqrt{2n}}}_{x}\cdot\sqrt{2n}\Big)$$

$$= \sum_{\substack{x:\, a \le x \le b \\ x \in \frac{2\mathbb{Z}}{\sqrt{2n}}}} P(S_{2n} = x\sqrt{2n})$$



$$\sim \sqrt{\frac{2}{n}}\sum_{\substack{x:\, a\le x\le b \\ x\in\frac{2\mathbb{Z}}{\sqrt{2n}}}} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \xrightarrow{n\to\infty} \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$

□.

$Var(X_i) = E[X_i^2] - (E[X_i])^2 = 1 - 0^2 = 1$.

$Var(S_n) = Var\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n Var(X_i) = n$.

So the standard deviation of S_n is $\sqrt{n}$.

So the "typical size" of $|S_n|$ is $\sim \sqrt{n}$.

the "typical size" of S_n^2 is n.

$Var(X) = E[(X-E(X))^2]$ $\quad E[|X - E[X]|]$

Let's see why the $\sqrt{2\pi}$ is there

we should have

$$P\left(-\infty \leq \frac{S_n}{\sqrt{n}} \leq \infty\right) \longrightarrow \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx.$$

$$\|$$

$$1.$$

So this should be 1.

Let's compute $\int_{-\infty}^{\infty} e^{-x^2/2}\,dx$. I don't know how.

I do know how to compute

$$\left(\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2/2 - y^2/2}\,dx\,dy$$

$$= \int_0^{\infty}\int_0^{2\pi} e^{-r^2/2}\cdot r\,d\theta\,dr = 2\pi\int_0^{\infty} re^{-r^2/2}\,dr$$

$$= 2\pi\int_0^{\infty} e^{-u}\,du = 2\pi.$$

So $\int_{-\infty}^{\infty} e^{-r^2/2}\,dr = \sqrt{2\pi}.$