# MATH 541, FALL 2023, HOMEWORK 1

Due Friday, September 8, 2023 at 5pm Eastern time on Gradescope.

You should make sure you know how to do Problems 1.1–1.4 in the textbook, but you do not have to write up solutions to turn in.

**Problem 1.** Exhibit an irreducible Markov chain on at least five states of period 3. Explain your example.

**Problem 2.** Give a proof or counterexample to the following statement: if $X_0, X_1, X_2, \ldots$ is a Markov chain on a finite state space $\mathcal{X}$, $f : \mathcal{X} \to \mathcal{X}$ is a function, and we define $Y_i = f(X_i)$, then $Y_0, Y_1, Y_2, \ldots$ is also a Markov chain.

**Problem 3.** Consider simple random walk on the vertices of a graph $G = (V, E)$. Give a formula for the invariant measure.

**Problem 4.** Follow the steps outlined in Problem 1.20 of the textbook to give a proof of the Perron–Frobenius theorem. Some extra hints/notes:

- Hint for (c): think of $\mathbf{A}\bar{v}$ as the $\bar{v}$ in part (a).
- Hint for (d): show that $\mathbf{A}|\bar{v}_1 - \bar{v}_2| > |\mathbf{A}(\bar{v}_1 - \bar{v}_2)|$ and then use part (c).
- In part (g), you can assume that $\mathbf{B}$ is a *principal* submatrix of $\mathbf{A}$. Hint: let $\beta$ be what you get for $\alpha$ when $\mathbf{A}$ is replaced by $\mathbf{B}$. Let $\bar{w}$ be the vector obtained in part (d) when $\mathbf{A}$ is replaced by $\mathbf{B}$, so $\bar{w} > 0$ and $\mathbf{B}\bar{w} = \beta\bar{w}$. So $\bar{w}$ is a vector in $\mathbb{R}^{n-1}$: can you think of a way to make it into a vector $\bar{z}$ of length $n$ such that $\mathbf{A}z > \beta z$?
- Hint for (h): one way to do this is to use the formula for the determinant

  $$\det \mathbf{M} = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{j=1}^{n} m_{j,\sigma(j)}.$$

  Another way is to note that $\lambda$ appears $n$ times in the matrix $\mathbf{A} - \lambda\mathbf{I}$, so the derivative has $n$ terms. Simplify the $i$th term in the expression for the derivative by using the cofactor expansion about the $i$th row/column.