Math 790 Regularity Structures and the KPZ equation

No lecture on MLK Day. Make-up lecture Friday 1/16.

For credit: do one "DO" exercise and turn in each Monday.

· Find notes and references on the course webpage.

Email me to receive email announcements

KPZ Equation: $h = h(t,x)$

$$\partial_t h = \frac{1}{2}\Delta h + \frac{\beta}{2}(\partial_x h)^2 + \xi.$$

$t \in \mathbb{R}$, $x \in \mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$.

$\beta \in \mathbb{R}$.

ξ: space-time white noise.

Gaussian process.

$$\mathbb{E}[\xi(t,x)] = 0.$$

$$\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y).$$

i.e. if $f \in L^2(\mathbb{R}^2)$, then

$$\mathbb{E}\left[\left(\int\int f(t,x)\xi(t,x)\,dx\,dt\right)^2\right]$$

$$= \int\int\int\int f(t,x) f(s,y)\,\mathbb{E}[\xi(t,x)\xi(s,y)]\,dx\,dy\,ds\,dt$$

$$= \int\int f(t,x)^2\,dx\,dt.$$

Cole-Hopf transform for KPZ

$$\partial_t h = \frac{1}{2}\Delta h + \frac{1}{2}(\partial_x h)^2 + \xi.$$

if we define $\phi = e^h$, then by using the chain rule, we can "see" that ϕ solves

$$\partial_t \phi = \frac{1}{2}\Delta\phi + \phi\xi. \quad \text{(MSHE) or (PAM)}$$

This equation can be solved using Itô integration. In particular: Duhamel formula $\Rightarrow$

$$\phi_t(x) = G_t * \phi_0(x) + \int_0^t G_{t-s} * \underbrace{(\phi_s \xi_s)}_{\text{Itô product.}}\, ds.$$

Set up a fixed-point problem to solve this Itô integral equation. Then define

$$h(t,x) = \log \phi(t,x).$$

"Cole-Hopf solution to KPZ".

"Whatever the physicists mean by the solution to KPZ, it must be this."

This is the solution that should arise as the limit of physical models.

So: what is the problem?

$\rightarrow$ Use of the chain rule is <u>not</u> justified. We have an Itô product, so we should use Itô's formula.

write $\xi(t,x) = dW(t,x)$. Then

$$dh = \left(\frac{1}{2}\Delta h + \frac{1}{2}(\partial_x h)^2\right)dt + dW - \infty$$

Itô correction

More precisely, if we regularize the noise $\rho^\varepsilon = \varepsilon^{-1}\rho(\varepsilon^{-1}\cdot)$, for a mollifier ρ, and then define $dW^\varepsilon = \rho^\varepsilon * dW$, then $d[W^\varepsilon]_t = \frac{\|\rho\|_{L^2}^2}{\varepsilon}$ and so

$$dh^\varepsilon = \left(\tfrac{1}{2}\Delta h^\varepsilon + \tfrac{1}{2}(\partial_x h^\varepsilon)^2\right)dt + dW^\varepsilon - \frac{\|\rho\|_{L^2}^2}{2\varepsilon}dt.$$

So in what sense is the CH solution really "solving" the KPZ equation?

→ We want a good approximation theory for the equation. Using CH approach to approximate requires a Cole-~~Hopf~~ transform at the level of the approximation.

For e.g. exclusion processes, this can be done with some ingenuity. "discrete CH transform" Bertini-Giacomin '97.

So we really want to try to solve the KPZ equation. Let's try naively

$$h_t(x) = G_t * h_0(x) + \int_0^t G_{t-s} * \xi_s \, ds + \frac{1}{2}\int_0^t G_{t-s} * (\partial_x h_s)^2 \, ds.$$

Let's try to iterate based on the linear equation. Let me ignore the i.c. for now.

$$G_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}.$$

$$h_t^{(0)}(x) = \int_0^t G_{t-s} * \xi_s \, ds.$$

$$h^{(0)}_t(x) = \int_0^t G_{t-s} * \xi_s \, ds.$$

$$h^{(1)}_t(x) = h^{(0)}_t(x) + \frac{1}{2}\int_0^t G_{t-s} * (\partial_x h^{(0)}_s)^2 \, ds$$

$$h^{(j)}_t(x) = h^{(0)}_t(x) + \frac{1}{2}\int_0^t G_{t-s} * (\partial_x h^{(j-1)}_s)^2 \, ds$$

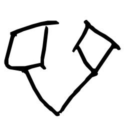
We will use a graphical notation for iterated integrals of ξ.

$\bullet = \xi.$

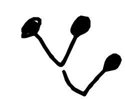
space-time convolution

 $= G \circledast \Box$

 $= \partial_x G \circledast \Box$

 $=$ () () $= (\partial_x G \circledast \Box)(\partial_x G \circledast \Diamond).$

In particular: $\partial_x h^{(0)} =$  "lollipop".

$\boxed{\text{DO}}$: write down the integral formula for .

<u>Rules</u>: · Regularity of ξ is $-\frac{3}{2}-$

- Product of distributions of regularity α and β has regularity $\min\{\alpha, \beta, \alpha+\beta\}$.
- $G\circledast$ improves regularity by 2.
- Two distributions can be multiplied if the sum of their regularities is strictly positive.
- In many cases, we can make sense of the product of distributions built from finite powers of white noise even when the sum of regularities is not positive.

$$h = \mathsf{I} + G\circledast(\partial_x h)^2.$$

$$g^{(1)} = h - \mathsf{I}. \qquad h = \mathsf{I} + g^{(1)}.$$

$$\Rightarrow g^{(1)} = G\circledast(\partial_x g^{(1)})^2 + 2\, G\circledast(\mathsf{I}\,\partial_x g^{(1)}) + G\circledast \mathsf{V}.$$

ill-defined product, but can be understood after renormalization.

I has regularity $-\frac{1}{2}-$.

$\boxed{\text{DO}}$: $\mathbb{E}[\mathsf{I}(t,x)\,\mathsf{I}(s,y)] = G_{t-s}(x-y)$ + smooth.

$$\mathbb{E}[\mathsf{V}(t,x)\,\mathsf{V}(s,y)]$$

$$\overset{\text{Isserlis}}{=} \mathbb{E}[\mathsf{V}(t,x)]\,\overbrace{\mathbb{E}[\mathsf{V}(s,y)]}^{\mathbb{E}[\mathsf{I}(s,y)^2] = \infty} + 2\underbrace{\left(\mathbb{E}[\mathsf{I}(t,x)\,\mathsf{I}(s,y)]\right)^2}_{\text{finite}}.$$

ignore this term.

"Wick product".

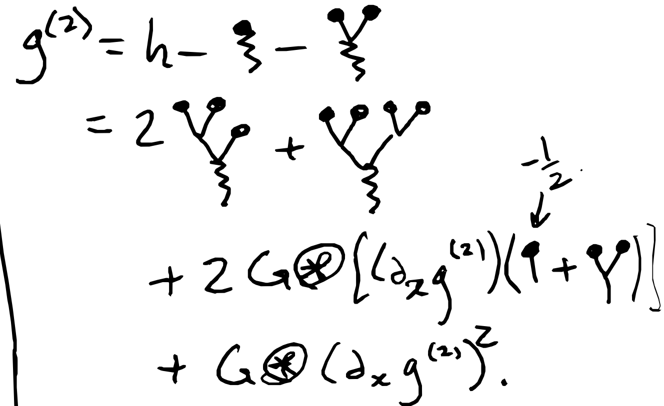
<u>Isserlis theorem:</u> if

$Z_1, \dots, Z_{2n}$ are mean-0 Gaussian, then

$$\mathbb{E}[Z_1 \cdots Z_{2n}] = \sum_{\pi} \prod_{(i,j)\in\pi} \mathbb{E}[Z_i Z_j]$$

π is a perfect matching of $\{1, \dots, 2n\}$.

Using some probability, we can define  after renormalization as an element of regularity $-1-$.



problem: regularity stops improving once terms have positive regularity. $g^{(i)}$ will never have regularity better than $3/2-$ because at every stage, you have a term like

On the other hand, we always want to multiply $\underset{1/2-}{\partial_x g^{(i)}}$  $-\frac{1}{2}-$

So what we need is a more systematic way of interpreting ill-defined products, one that allows us to take products of objects that we don't have an explicit Gaussian understanding of.
This will be provided by rough path theory, which developed into regularity structures