Lecture 2.

Lecture recordings are available at
<course website>/recordings.html.

Reminder: first HW due today.

Last time: to solve KPZ we want to multiply distributions/functions such that the sum of the regularities is not necessarily positive.

Today: rough path theory "how to do this for functions of one variable".

Actually rough paths are enough to solve KPZ. (Hairer 2013)

Young integration. (1936)

Interested in the Riemann-Stieltjes integral $\int_0^1 f(t)\, dg(t)$. "=" $\int_0^1 f(t) g'(t) dt$

This should (presumably) be the limit of Riemann sums

$$\lim \sum_{i=1}^{N} f(x_{i-1})(g(x_i) - g(x_{i-1}))$$

(left Riemann sum)

or

$$\lim \sum_{i=1}^{N} f(x_i)(g(x_i) - g(x_{i-1})).$$

(right Riemann sum)

So in particular, these limits should be the same.

The difference of the left & right sums is $[f \in C^\alpha, g \in C^\beta]$

$$\left| \sum_{i=1}^{N} (f(x_i) - f(x_{i-1}))(g(x_i) - g(x_{i-1})) \right|$$

$$\leq \sum_{i=1}^{N} \underbrace{|f(x_i) - f(x_{i-1})|}_{1/N} \cdot |g(x_i) - g(x_{i-1})|$$

$$\leq \sum |x_i - x_{i-1}|^{\alpha+\beta} \|f\|_{C^\alpha} \|g\|_{C^\beta}.$$

$|x_i - x_{i-1}| \sim \frac{1}{N}$, so this is like

$$N \cdot \frac{1}{N^{\alpha+\beta}} \to 0 \quad \text{iff } \alpha+\beta > 1.$$

Thm. If $\alpha+\beta > 1$, $f \in C^\alpha$, $g \in C^\beta$, then the Young integral

$$\int_0^T f(t)\,dg(t) \text{ is well-defined,}$$

and it obeys the estimate

$$\left| \int_0^T f(t)\,dg(t) - f(s)(g(T) - g(0)) \right| \lesssim \|f\|_{C^\alpha} \|g\|_{C^\beta} T^{\alpha+\beta}.$$

for any $s \in (0,T)$.

Beyond Young integration?

Itô integration (195x?)

We can define (B_t std BM)

$$\int_0^T B_t\,dB_t \quad \text{as a limit in}$$

probability of Left Riemann sums.

Note: this does not fall into the framework of Young integration because $B \in C^\alpha$, $\alpha < 1/2$.

In particular, $\int^T B_t\,dB_t$ is not a continuous function of the path B if we put the C^α topology on B. To see that it is not continuous, think about shifting integrand B slightly in time to turn left RS into right RS.

and we know that the right Riemann sum is converging to the Itô integral plus twice the QV. (or midpoint RS converges to Stratonovich)

Itô theory: stabilize the integration theory by imposing probabilistic conditions, i.e. demand that the integral be a (local) martingale.

But we would like a pathwise theory. For $\int_0^t B_t\, dB_t$, no choice but to define it probabilistically.

But what about (assume $f \in C^2$)

$$\int_0^t f(B_t)\, dB_t \ ?$$

Itô integration lets us define this as well. But can we do it pathwise?

$$\int_0^T f(B_t)\, dB_t = \sum_{i=1}^N \int_{t_{i-1}}^{t_i} f(B_t)\, dB_t$$

$$= \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \Big[f(B_{t_{i-1}}) + \underbrace{f'(B_{t_{i-1}})}_{A}\underbrace{(B_t - B_{t_{i-1}})}_{B} + \frac{1}{2} \underbrace{f''(B_{t_i^*})(B_t - B_{t_{i-1}})^2}_{C} \Big] dB_t$$

$\hookrightarrow t_i^* \in [t_{i-1}, t_i]$

Typical size of A is $|t_i - t_{i-1}|^\alpha$ not going to 0.

Typical size of B is $|t_i - t_{i-1}|^{2\alpha}$.

$\boxed{\text{DO}}$: check this using Itô integration.

Typical size of C is $|t_i - t_{i-1}|^{3\alpha}$.

Point: A and B are not negligible. but C is negligible since we can take $\alpha > 1/3$.

What this means is that if we define

$$\mathbb{B}_{s,t} := \int_s^t (B_r - B_s)\, dB_r, \quad \text{"iterated integral".}$$

then we can write

$$\int_0^T f(B_t)\, dB_t \approx \sum_{i=1}^N \Big\{ f(B_{t_{i-1}}) \underbrace{(B_{t_i} - B_{t_{i-1}})}_{= B_{t_{i-1}, t_i}} + f'(B_{t_{i-1}})\, \mathbb{B}_{t_{i-1}, t_i} \Big\}.$$

<u>Theorem.</u> This sum converges as the mesh size goes to 0, and the resulting "rough integral" is <u>continuous</u> as a function of $(f, f', B, \mathbb{B})$ where B has the topology of C^α, and $\mathbb{B}$ has the topology generated by

$$\sup_{t \neq s} \frac{|\mathbb{B}_{s,t}|}{|t-s|^{2\alpha}}.$$

$$\begin{tikzcd}
 & (B, \mathbb{B}) \arrow[dr, "\text{cts.}"] & \\
 & B \arrow[u, "\text{msbl}"] \arrow[r, "\text{msbl}"'] & \int_0^T f(B_t)\, dB_t \\
\omega \in \Omega \arrow[ur] & \rotatebox{90}{$\in$}\ C^\alpha &
\end{tikzcd}$$

Observe: for this to work, we don't actually need f to be a fn of B_t. Rather, it is sufficient to replace $f(B_t)$ by some Y_t s.t. there exists a Y'_t such that

$$Y_t - Y_s = Y'_s B_{s,t} + R_{s,t}, \text{ where}$$

$|R_{s,t}| \lesssim |t-s|^{2\alpha}$. In this case, we say that Y is controlled by B with Gubinelli derivative Y'.

(Note: Y' has nothing to do with the ordinary derivative, and may not even be unique.)

Now let's try to make this precise

Definition. Let $\alpha \in (1/3, 1/2]$, and let V be a Banach space (e.g. $\mathbb{R}^d$)

An α-Hölder rough path over V is a pair $\big(X : [0,T] \to V,$

$$\mathbb{X} : [0,T]^2 \to V \otimes V\big)$$

such that

$$\|X\|_\alpha := \sup_{s<t} \frac{|X_{s,t}|}{|t-s|^\alpha} < \infty \qquad X_{s,t} = X_t - X_s.$$

and

$$\|\mathbb{X}\|_{2\alpha} := \sup_{s<t} \frac{|\mathbb{X}_{s,t}|}{|t-s|^{2\alpha}} < \infty.$$

and such that

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = X_{s,u} \otimes X_{u,t}.$$

(Chen's relation).

Why Chen's relation?

$$\mathbb{X}_{s,t} \text{ "=" } \int_s^t X_{s,r} \otimes dX_r.$$

$$\int_s^t X_{s,r} \otimes dX_r - \int_s^u X_{s,r} \otimes dX_r$$

$$- \int_u^t X_{u,r} \otimes dX_r$$

$$= \int_s^t (X_r - X_s) \otimes dX_r - \int_s^u (X_r - X_s) \otimes dX_r$$

$$- \int_u^t (X_r - X_u) \otimes dX_r.$$

$$= - X_s \otimes (X_t - X_s)$$

$$+ X_s \otimes (X_u - X_s)$$

$$+ X_u \otimes (X_t - X_u)$$

$$= (X_u - X_s) \otimes (X_t - X_u).$$

$\boxed{\text{Do}}$: show that $(X, \mathbb{X})$ if $X_0 = 0$ can be recovered from the path $t \mapsto (X_{0,t}, \mathbb{X}_{0,t})$ along w/ Chen's relation.

Rough paths are <u>integrators</u>. Now we want to define the space of integrands.

Given a function $X \in C^\alpha([0,T], V)$, we say that a $Y \in C^\alpha([0,T], \bar{W})$ is <u>controlled by X</u> if there exists a $Y' \in C^\alpha([0,T], \mathcal{L}(V, \bar{W}))$ such that the remainder term given by

$$Y_{s,t} = Y_s' X_{s,t} + R^Y_{s,t} \quad \text{satisfies}$$

$\|R^Y\|_{2\alpha} < \infty$. (e.g. if $Y_s = f(X_s)$ then take $Y_s' = f'(X_s)$)

This defines the space of **controlled rough paths** $(Y, Y') \in \mathcal{D}_X^{2\alpha}([0,T], W)$.

We call any such Y' a **Gubinelli derivative** of Y w.r.t. X. Equip this space with the seminorm

$$\|Y, Y'\|_{X, 2\alpha} := \|Y'\|_\alpha + \|R^Y\|_{2\alpha}.$$

Next time: prove this theorem using "sewing lemma", which is a precursor to the "reconstruction theorem" in regularity structures

Then we can define the **rough integral**

$$\int_s^t Y \, d(X, \mathbb{X}) = \lim \sum_{i=1}^N \left(Y_{t_{i-1}} X_{t_{i-1}, t_i} + Y'_{t_{i-1}} \mathbb{X}_{t_{i-1}, t_i} \right).$$

Theorem. This limit exists, and we have the bound

$$\left| \int_s^t Y_r \, dX_r - Y_s X_{s,t} - Y'_s \mathbb{X}_{s,t} \right| \le C \left(\|X\|_\alpha \|R^Y\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha} \|Y'\|_\alpha \right) \cdot |t-s|^{3\alpha}.$$

Moreover, the map from $\mathcal{D}_X^{2\alpha}([0,T], \mathcal{L}(V, W))$ to $\mathcal{D}_X^{2\alpha}([0,T], W)$ given by $(Y, Y') \longmapsto \left(\int_0^\bullet Y_r \, dX_r, Y \right)$ is cts. and satisfies a certain bound