Lecture 3: Sewing lemma

$\alpha > 1/3$

Recall. A rough path is a pair $(X, \mathbb{X})$ s.t. $X \in C^\alpha$, $\mathbb{X} \in$ "$\mathcal{C}^{2\alpha}$" in the sense that $\sup_{s<t} \frac{|\mathbb{X}_{s,t}|}{|t-s|^{2\alpha}} < \infty$

and Chen's relation holds, ie

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = X_{s,u} \otimes X_{u,t}$$

Think: "$\mathbb{X}_{s,t} = \int_s^t (X_r - X_s)\, dX_r$"

but in reality the r.h.s might not be defined, and so really we think of the left side as a definition of the right side.

Recall. $Y \in C^\alpha$ is a rough path controlled by X if there is a "Gubinelli derivative" Y' s.t.

$$Y_{s,t} = Y'_s X_{s,t} + R^Y_{s,t}, \text{ where}$$

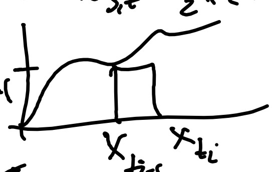
$$\sup_{s<t} \frac{|R^Y_{s,t}|}{|t-s|^{2\alpha}} < \infty.$$

Rough integral:

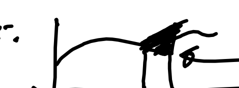
$$\int_s^t Y\, dX = \lim \sum_{i=1}^{N} \left(Y_{t_{i-1}} X_{t_{i-1}, t_i} + Y'_{t_i} \mathbb{X}_{t_{i-1}, t_i} \right)$$

special case: $X_s = s$. $\mathbb{X}_{s,t} = \frac{1}{2}(t-s)^2$

$Y_{t_{i-1}} X_{t_{i-1}, t_i}$:



second term:



add area of this triangle

Theorem. The limit exists, and we have the bound

$$\left|\int_s^t Y_r\,dX_r - Y_s X_{s,t} - Y'_s \mathbb{X}_{s,t}\right| \le C\left(\|X\|_\alpha \|R^Y\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha}\|Y'\|_\alpha\right) \cdot |t-s|^{3\alpha}$$

Moreover, for fixed $(X,\mathbb{X})$, the map

$$\mathcal{D}_X^{2\alpha} \longrightarrow \mathcal{D}_X^{2\alpha}$$ given by

$$(Y,Y') \longmapsto (Z,Z') = \left(\int_0^\cdot Y_r\,dX_r,\ Y\right)$$

is continuous linear map and

$$\|Z,Z'\|_{X,2\alpha} \le \|Y\|_\alpha + \|Y'\|_\infty \|X\|_{2\alpha} + CT^\alpha\left(\|X\|_\alpha \|R^Y\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha} \cdot \|Y'\|_\alpha\right)$$

__Warning__: you would like to say that the rough integral is bilinear in (Y,X). This doesn't really make sense to say because the Banach space $\mathcal{D}_X^{2\alpha}$ that (Y,Y') lives in depends on X. The space of allowed integrands depends on the integrator.

Let's try to prove this. The key is the "__sewing lemma__".

Sewing lemma. In Young integration, if "$Z_{s,t} = \int_s^t Y \, dX$", then we write

$$Z_{s,t} = \underbrace{Y_s X_{s,t}}_{\Xi_{s,t}} + o(|t-s|) \quad (*)$$

$\Xi_{s,t}$ "local approximation of the integral"

To define the integral, we want to go backwards. Given $\Xi_{s,t}$, we want to find $Z_{s,t}$ s.t. $(*)$ holds and

$Z_{s,t} = Z_{s,u} + Z_{u,t}$. (i.e. $\exists Z$ s.t. $Z_{s,t} = Z_t - Z_s$)

We call "sewing" the step from Ξ to $Z := \mathcal{I}\Xi$. ($\mathcal{I}$ is the sewing map.)

Let's start by defining a space for Ξ.

$C_2^{\alpha,\beta}$: space of functions Ξ on $\{s \le t\}$ s.t. $\Xi_{t,t} = 0$ and

$$\|\Xi\|_{\alpha,\beta} := \|\Xi\|_\alpha + \|\delta\Xi\|_\beta$$

where $\|\Xi\|_\alpha = \sup_{s<t} \frac{|\Xi_{s,t}|}{|t-s|^\alpha}$

$$\delta\Xi_{s,u,t} = \Xi_{s,t} - \Xi_{s,u} - \Xi_{u,t}$$

$$\|\delta\Xi\|_\beta = \sup_{s<u<t} \frac{|\delta\Xi|_{s,u,t}}{|t-s|^\beta}.$$

Informally: if $\beta > 1$, then Ξ is almost additive, i.e. $\exists \hat{\Xi}$ w/ $\delta\hat{\Xi} = 0$ s.t. $\hat{\Xi}_{s,t} \approx \Xi_{s,t}$ when $|s-t| \ll 1$.

<u>Lemma</u> 2. (Sewing lemma) Let $0<\alpha\le 1<\beta<\infty$. Then there is a unique cts linear map

$$\underset{\sim}{\mathcal{I}} : \mathcal{C}_2^{\alpha,\beta}([0,T]) \longrightarrow \mathcal{C}^{\alpha}([0,T])$$

s.t. $(\underset{\sim}{\mathcal{I}}\Xi)_0 = 0$. and

$$|(\underset{\sim}{\mathcal{I}}\Xi)_{s,t} - \Xi_{s,t}| \le C\|\delta\Xi\|_\beta |t-s|^\beta$$

$[\delta\Xi_{s,u,t} = \Xi_{s,t} - \Xi_{s,u} - \Xi_{u,t}]$.

Before proving, let's apply it for Young integration. Define $\Xi_{s,t} = Y_s X_{s,t}$

then
$$\delta\Xi_{s,u,t} = Y_s X_{s,t} - Y_s X_{s,u} - Y_u X_{u,t}$$
$$= \underbrace{(Y_s - Y_u)}_{\le |u-s|^\alpha} \cdot \underbrace{X_{u,t}}_{|t-u|^\alpha}.$$

if $\boxed{2\alpha>1}$ (Young case) then sewing lemma holds.

Proof. Since $\mathcal{I}$ will be defined as a linear map, sufficient to check boundedness.

$I = \mathcal{I}\Xi$. will be constructed by setting $I_0 = 0$ and then building $I_{s,t} = I_t - I_s$. Uniqueness is automatic, since two paths I & $\hat{I}$ satisfying the conclusion would have

$$|(I-\hat{I})_{s,t}| \lesssim |(I-\Xi)_{s,t}| + |(\hat{I}-\Xi)_{s,t}|$$
$$\lesssim |t-s|^\beta.$$

and since $\beta>1$ this means $I = \hat{I}$.

In fact,
$$\left|I_{s,t} - \sum_{i=1}^{N} \Xi_{t_{i-1},t_i}\right|$$
$$= \left|\sum_{i=1}^{N} (I_{t_{i-1},t_i} - \Xi_{t_{i-1},t_i})\right|$$
$$\to 0 \text{ as } N\to\infty.$$

Existence. Fix $[s,t] \subseteq [0,T]$.

Let P_n be the level-n dyadic partition of $[s,t]$. (having 2^n intervals of length $2^{-n}(t-s)$)

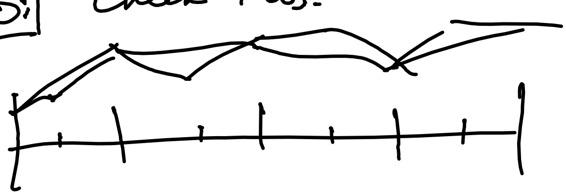
Define $I^0_{s,t} = \Xi_{s,t}$.

and then define

$$I^{n+1}_{s,t} = \sum_{[u,v]\in P_n} \Xi_{u,v}$$

$$= I^n_{s,t} - \sum_{[u,v]\in P_n} \delta \Xi_{u,\frac{u+v}{2},v}.$$

$\boxed{\text{DO:}}$ Check this.



So: $|I^n_{s,t} - I^{n+1}_{s,t}| \le 2^{n(1-\beta)}|t-s|^\beta \|\delta\Xi\|_\beta$.

this is summable in n. so

$I^n_{s,t} \longrightarrow$ something we call $I_{s,t}$ as $n \to \infty$.

in fact, $|I^n_{s,t} - I_{s,t}| \le C\|\delta\Xi\|_\beta 2^{(1-\beta)n}|t-s|^\beta$.

C independent of s,t.

in particular,

$$|\Xi_{s,t} - I_{s,t}| \le C\|\delta\Xi\|_\beta |t-s|^\beta$$

that's the bound that we wanted.

Still need to check additivity.

if $T=1$, $u = \frac{s+t}{2}$, then $I^{n+1}_{s,t} = I^n_{s,u} + I^n_{u,t}$.

$[s,t] = 2^{-k}[l, l+1]$ for some integer l.

take $n \to \infty$.

To extend this to general intervals $[s,t]$, approximate $[s,t]$ by an "efficient" union of dyadic intervals (at most one interval at each scale) so that the error is summable.

Let's check the conditions of the lemma for rough integrals.

$$\Xi_{s,t} = Y_s X_{s,t} + Y'_s \mathbb{X}_{s,t}.$$

$$\delta \Xi_{s,u,t} = -R^Y_{s,u} X_{u,t} - Y'_{s,u} \mathbb{X}_{u,t}.$$

$\boxed{\text{TODO}}$: check this using Chen's relation.

$$\frac{|\delta \Xi_{s,u,t}|}{|t-s|^{3\alpha}} \le \frac{\overbrace{|R^Y_{s,u}|}^{\le |s-u|^{2\alpha}} \cdot \overbrace{|X_{u,t}|}^{|t-u|^{\alpha}} + \overbrace{|Y'_{s,u}|}^{\le |u-s|^{\alpha}} \cdot \underbrace{|\mathbb{X}_{u,t}|}_{|t-u|^{2\alpha}}}{|t-s|^{3\alpha}}$$

$< \infty$

So if $3\alpha > 1$, then the sewing lemma applies and we can define the rough integral.

<u>Comment:</u> if $\alpha < 1/3$, then still define rough integrals, but you need more iterated integrals to be specified (N iterated integrals for $\alpha \in \left(\frac{1}{N+2}, \frac{1}{N+1}\right)$).

FRIDAY 1/16: <u>Physics 235</u>. <u>10:05-11:20</u>.
(zoom link is the same)