Lecture 4: Intro to Regularity Structures

Recall. if W is a path then $(Y, Y') \in \mathcal{D}_W^{2\alpha}$ (controlled rough path) means there is a "Taylor expansion"

$$Y_t = Y_s + Y'_s W_{s,t} + O(|t-s|^{2\alpha}).$$

Recall. in the usual case of smooth functions, we can write

$$f_t = f_s + \sum_{\ell=1}^{k} \frac{1}{\ell!} f_s^{(\ell)} (t-s)^\ell + o(|t-s|^k).$$

A regularity structure is a further generalization of this situation.

Ingredients:

① basis of "Taylor polynomials"
- vector space T.
- graded vector space.

$$T = \bigoplus_{\alpha \in A} T_\alpha, \quad \text{for some } A \subseteq \mathbb{R}.$$

"degree α"

For rough paths, $A = \{0, \alpha\}$.

For polynomials, $A = \{0, 1, 2, \dots\}$.

The point is that an element of T_α should represent an object that vanishes to order α at $t = s$.

e.g. $|W_{s,t}| \lesssim |t-s|^\alpha$.

or $|(t-s)^\ell| \lesssim |t-s|^\ell$.

Taylor expansions are good for approximating functions locally about a single point. But maybe we want to understand a function around a different point. So we need to "re-expand" the Taylor expansion about x_0 around a different point x_1.

$$\underbrace{(x-x_0)^m}_{\text{original basis element}} = \sum_{k+l=m} \underbrace{\frac{m!}{k!\,l!}(x_1-x_0)^k}_{\text{coefficients}} \cdot \underbrace{(x-x_1)^l}_{\text{new basis element}}.$$

Similarly, for controlled rough paths, can write $W_{s_0,s} = W_{s_0,s_1}\cdot 1 + 1\cdot W_{s_1,s}$.

So if originally, we have the local expansion about s_0

$$\begin{aligned}
Y_s &= Y_{s_0}\cdot 1 + Y'_{s_0}\cdot W_{s_0,s} \\
&= Y_{s_0}\cdot 1 + Y'_{s_0}\left(W_{s_0,s_1}\cdot 1 + 1\cdot W_{s_1,s}\right) \\
&= \overbrace{\left(Y_{s_0} + Y'_{s_0} W_{s_0,s_1}\right)}^{\approx Y_{s_1}}\cdot 1 + Y'_{s_0}\cdot W_{s_1,s}.
\end{aligned}$$

which is a local expansion about s_1.

Definition. A regularity structure $\mathcal{T} = (T, G)$ comprises the following:

- a "structure space" ("model space" in the original paper) which is a graded vector space $T = \bigoplus_{\alpha \in A} T_\alpha$ with each T_α a Banach space and index set $A \subseteq \mathbb{R}$ locally finite and bounded below.

Elements of T_α are said to have degree α, write $\deg \tau = \alpha$ for $\tau \in T_\alpha$. Given $\tau \in T$, we write $\|\tau\|_\alpha$ for the norm of the component in T_α.

- "structure group" G of continuous linear maps on T s.t. for every $\Gamma \in G$, every $\alpha \in A$, and every $\tau \in T_\alpha$, we have

$$\Gamma \tau_\alpha - \tau_\alpha \in T_{<\alpha} := \bigoplus_{\beta < \alpha} T_\beta .$$

A *sector* of $\mathcal{T}$ is a linear subspace $V = \bigoplus_{\alpha \in A} V_\alpha \subseteq T$, $V_\alpha \subseteq T_\alpha$, s.t. V is invariant under G and $(V, G|_V)$ is a regularity structure itself.

Polynomial regularity structure:

$T = \mathbb{R}[\underline{X}_1, \dots, \underline{X}_d]$, where $\underline{1}$ is the identity in this space

$\underline{X}_1, \dots, \underline{X}_d$ are d commuting indeterminants. Then T_α comprises homogeneous polynomials of degree $\alpha \in \mathbb{N}$.

$G \cong (\mathbb{R}^d, +)$, acting on T

via
$$\Gamma_h P(\underline{X}) = P(\underline{X}_1 + h_1 \underline{1}, \dots, \underline{X}_d + h_d \underline{1}).$$

Let's work out what this means for $d=1$, degree 2 case so restrict to $T = \langle \underline{1}, \underline{X}, \underline{X}^2 \rangle$.

If $f \in C^{2+\gamma}$, $\gamma \in (0,1)$, could represent the Taylor expansion of f by the map $y \longmapsto f(y)\underline{1} + f'(y)\underline{X} + \frac{1}{2} f''(y) \underline{X}^2$.

Note: $\underline{X}$ represents "$x-y$".

$\underline{X}^2$ represents "$(x-y)^2$".

the meaning depends on y! if we want to re-expand f around a different point z, then we would apply Γ_{z-y} and get...

$$y \longmapsto f(y)\underline{1} + f'(y)\left(\underline{X} + (z-y)\underline{1}\right) + \frac{1}{2}f''(y)\left(\underline{X} + (z-y)\underline{1}\right)^2$$

[now "$\underline{X} = x - z$" so to match with original expansion write "$\underline{X} + (z-y)\underline{1}$".]

$$= \left[f(y) + f'(y)(z-y) + \frac{1}{2}f''(y)(z-y)^2\right]\underline{1} + \left[f'(y) + f''(y)(z-y)\right]\underline{X} + \frac{1}{2}f''(y)\underline{X}^2.$$

Model. associate to the abstract elements of T_α a function/distribution that "vanishes to order α" around a given point x.

Given a test function φ on $\mathbb{R}^d$, we define

$$\varphi_x^\lambda(y) = \lambda^{-d}\varphi(\lambda^{-1}(y-x))$$

Let $\mathcal{B}_r = \{$ test functions φ supported on $B(0,1)$ s.t. $\|\varphi\|_{C^r} \le 1 \}$

Defn. Given a RS $\mathcal{T} = (T, G)$ and a $d \geq 1$ a **model** $M = (\Pi, \Gamma)$ for $\mathcal{T}$

comprises

$$\Pi : \mathbb{R}^d \to \mathcal{L}(T, \mathcal{D}'(\mathbb{R}^d))$$

distns on $\mathbb{R}^d$

$$\Gamma : \mathbb{R}^d \times \mathbb{R}^d \to G$$

s.t. $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$ and

$\Pi_x \Gamma_{xy} = \Pi_y$. and moreover, if r is the smallest integer s.t. $r > |\min A| \geq 0$ then for any compact $K \subseteq \mathbb{R}^d$, $\gamma > 0$ $\exists\, C = C(K,\gamma)$ s.t. $\forall \tau \in T_\alpha$, $\alpha \le \gamma$, $\beta < \alpha$

$$|\langle \Pi_x \tau, \varphi_y^\lambda \rangle| \le C \lambda^\alpha \|\tau\|_\alpha.$$

$$\|\Gamma_{xy}\tau - \tau\|_\beta \le C|x-y|^{\alpha-\beta}\|\tau\|_\alpha.$$

Π is called "realization"

Γ is called "reexpansion".