Lecture 5.

Recall the definition of a model. Given a RS $\mathcal{T} = (T, G)$ (A = set of degrees) and $d \geq 1$, a model $M = (\Pi, \Gamma)$ comprises

$$\Pi : \mathbb{R}^d \to \mathcal{L}(T, \mathcal{D}'(\mathbb{R}^d))$$

$$\Gamma : \mathbb{R}^d \times \mathbb{R}^d \to G$$

s.t.

$$\left.\begin{aligned} &\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz} \\ \text{and } \quad &\Pi_x \Gamma_{xy} = \Pi_y . \end{aligned}\right\} \text{algc conditions}$$

Let r be the smallest integer s.t. $r > |\min A| \geq 0$. Then for any compact $K \subseteq \mathbb{R}^d$, require

analytic conds

for all $\alpha \in A$, $\tau \in T_\alpha$

$$\left\{\begin{aligned} &|\langle \Pi_x \tau, \varphi_x^\lambda \rangle| \leq C \lambda^\alpha \|\tau\|_\alpha \\ &\|\Gamma_{xy}\tau \|_\beta \leq C |x-y|^{\alpha-\beta} \|\tau\|_\alpha . \end{aligned}\right.$$

$\|\Gamma_{xy}\tau - \tau\|_\beta$

$\forall \beta < \alpha$.

Note: space of models is not a linear space, but embeds into a linear space given by removing the constraints, and the analytic constraints make this a Fréchet space.

Example for polynomials.

Recall the basis of T

is $\underline{1}, \underline{X}, \underline{X}^2, \ldots$

Then
$$\Pi_x \underline{X}^k (y) = (y-x)^k$$
$$\Gamma_{xy} \underline{X}^k = (\underline{X} + (y-x)\underline{1})^k$$

Then ~~can~~ check:

$$\Pi_x(\Gamma_{xy} \underline{X}^k)(z).$$
$$= \Pi_x\left(\left(\underline{X} + (x-y)\underline{1}\right)^k\right)(z)$$
$$= (z-x+x-y)^k = (z-y)^k$$
$$= \Pi_y \underline{X}^k(z).$$

$\boxed{\text{DO:}}$ check one of the analytic conditions.

Modeled distributions.

Given a regularity structure $\mathcal{T}$ equipped with a model $M=(\Pi,\Gamma)$ over $\mathbb{R}^d$, the space $\mathcal{D}^\gamma_M$ of modeled distributions is given by the set of functions

$\underline{f}: \mathbb{R}^d \to T_{<\gamma}$ s.t. for every compact set K and every $\alpha<\gamma$ there is a $C<\infty$ s.t.

$$\|\underline{f}(x) - \Gamma_{xy}\underline{f}(y)\|_\alpha \le C|x-y|^{\gamma-\alpha}. \quad \forall x,y \in K.$$

Example. Say $f \in C^\gamma$ for some $\gamma \in (2,3)$. Then we could lift f to a modeled distn $\underline{f} \in \mathcal{D}^\gamma_M$ by.

$$\underline{f}(x) = f(x)\underline{1} + f'(x)\underline{X} + \tfrac{1}{2}f''(x)\underline{X}^2.$$

Let's check that $\underline{f}$ really does live in $\mathcal{D}^\gamma_M$.

$$\Gamma_{xy}\underline{f}(y) = f(y)\underline{1} + f'(y)\left(\underline{X} + (x-y)\underline{1}\right) + \tfrac{1}{2}f''(y)\left(\underline{X} + (x-y)\underline{1}\right)^2$$

$$= \left(f(y) + f'(y)(x-y) + \tfrac{1}{2}f''(y)(x-y)^2\right)\underline{1} + \left(f'(y) + f''(y)(x-y)\right)\underline{X} + \tfrac{1}{2}f''(y)\underline{X}^2.$$

So: $\underline{f}(x) - \Gamma_{xy}\underline{f}(y)$

$$= \overbrace{\left(f(x) - \left\{f(y) + f'(y)(x-y) + \tfrac{1}{2}f''(y)(x-y)^2\right\}\right)}^{|x-y|^\gamma}\underline{1}$$

$$+ \overbrace{\left(f'(x) - \left\{f'(y) + f''(y)(x-y)\right\}\right)}^{|x-y|^{\gamma-1}}\underline{X}$$

$$+ \tfrac{1}{2}\overbrace{\left(f''(x) - f''(y)\right)}^{|x-y|^{\gamma-2}}\underline{X}^2.$$

Theorem (Reconstruction Theorem)

Let $M = (\Pi, \Gamma)$ be a model for a regularity structure (T, G). ~~Let $\underline{f} \in \mathcal{D}^\gamma_M$.~~

Let ~~Let~~ $\gamma > 0$. Then there exists a unique linear map

$$\mathcal{R} = \mathcal{R}_M : \mathcal{D}^\gamma_M \to \mathcal{D}'(\mathbb{R}^d)$$

s.t. $\forall \underline{f} \in \mathcal{D}^\gamma_M$

$$\left| \langle \mathcal{R}\underline{f} - \Pi_x(\underline{f}(x)), \underset{\lambda^{-d}\varphi(\lambda^{-1}(y-x))}{\overset{\shortparallel}{\varphi^\lambda_x}} \rangle \right| \lesssim \lambda^\gamma.$$

uniformly over $\varphi \in B_r$.

For $\gamma < 0$, still valid except for uniqueness of $\mathcal{R}$.

Note: if Π_x takes values in qualitatively continuous functions, then

$$\mathcal{R}\underline{f}(x) = \Pi_x(\underline{f}(x))(x).$$

DO: check

Note for polynomials, $k \geq 1$

$$\Pi_x(\underline{X}^k)(x) = 0.$$

$$\text{"} (x-x)^k \text{"}$$

Extended example: Rough paths.

Let $\alpha \in (\frac{1}{3}, \frac{1}{2})$

RS: $T = T_{\alpha-1} \oplus T_{2\alpha-1} \oplus T_0 \oplus T_\alpha$.

$T_0 = \langle \underline{1} \rangle$.

$T_{\alpha-1} = \langle \dot{\underline{W}} \rangle$.

$T_{2\alpha-1} = \langle \dot{\underline{\mathbb{W}}} \rangle$.

$T_\alpha = \langle \underline{W} \rangle$.

Structure group $G \cong (\mathbb{R}, +)$.

acting on T by

$\Gamma_h \underline{1} = \underline{1}$

$\Gamma_h \dot{\underline{W}} = \dot{\underline{W}}$

$\Gamma_h \underline{W} = \underline{W} + h\,\underline{1}$

$\Gamma_h \dot{\underline{\mathbb{W}}} = \dot{\underline{\mathbb{W}}} + h\,\dot{\underline{W}}$

<u>Model</u>: $(W, \mathbb{W})$ α-Hölder rough path.

$\Pi_x \underline{1}(y) = 1$.

$\Pi_x \dot{\underline{W}}(y) = \partial_y W_y$ (as a distr)

$\Pi_x \underline{W}(y) = W_y - W_x$.

$\Pi_x \dot{\underline{\mathbb{W}}}(y) = \partial_y \mathbb{W}_{x,y}$.

Think: of $\mathbb{W}_{x,y} = \int_x^y (B_r - B_x)\, dB_r$.

"$\partial_y \mathbb{W}_{x,y} = (B_y - B_x)\, dB_y$".

Reexpansion map of the model:

$\Gamma_{xy} = \Gamma_{W_{y,x}}$.

<u>Intuition</u>:

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Intuition: Chen's relation $\Rightarrow$

$$W_{y,z} = W_{y,x} + W_{x,z} + W_{y,x} W_{x,z}$$

so $\partial_z W_{y,z} = \partial_z W_{x,z} + W_{y,x}\, \partial_z W_{x,z}$.

DO: check one (or several) of the alg. conditions on the model.

or one or more of the analytic conditions.

How do modeled distributions work?

example. controlled rough path. (Y, Y')

Gubinelli derivative

$$Y_x = Y_y + Y'_y W_{y,x} + o(|y-x|^{2\alpha}).$$

Lift Y to a modeled distribution by

$$\underline{Y}_y = Y_y \underline{1} + Y'_y \underline{W}.$$

Lets check that $\underline{Y}$ defines a modeled distn in $\mathcal{D}^{2\alpha}_{M}$.

$$\Gamma_{xy} \underline{Y}_y = Y_y \underline{1} + Y'_y (\underline{W} + W_{y,x} \underline{1})$$

$$= \underbrace{(Y_y + Y'_y W_{y,x})}_{Y_x + o(|x-y|^{2\alpha})} \underbrace{\underline{1}}_{\deg 0} + \underbrace{Y'_y}_{Y'_x + o(|x-y|^{\alpha})} \underbrace{\underline{W}}_{\deg \alpha}.$$