Lecture 6.

Finishing last time: how to use RS to set up rough integrals

Recall that the RS for rough paths has the basis

$$\underline{1}, \quad \underline{\dot{W}}, \quad \underline{\dot{\mathbb{W}}}, \quad \underline{W}$$

$$(0) \quad (\alpha-1) \quad (2\alpha-1) \quad (\alpha)$$

$\alpha \in (\frac{1}{3}, \frac{1}{2})$

Suppose we have a rough path $(W, \mathbb{W})$, and also a controlled rough path $(Y, Y') \in \mathcal{D}_W^{2\alpha}$.

Then we want to consider the rough integral

$$Z = \int_s^t Y \, dW \underset{s \sim t}{\approx} Y_s W_{s,t} + Y_s' \mathbb{W}_{s,t}.$$

we could represent this by the modeled distribution

$$\underline{\dot{Z}}_s = Y_s \underline{\dot{W}} + Y_s' \underline{\dot{\mathbb{W}}}.$$

Let's check the modeled distribution properties.

Recall. A function $\underline{f}: \mathbb{R}^d \to T$ is a modeled distribution of reg. γ if

$$\| \underline{f}(x) - \Gamma_{xy} \underline{f}(y) \|_\alpha \lesssim |x-y|^{\gamma-\alpha}$$

$$\Gamma_{r,s} \underline{\dot{Z}}_s = Y_s \underline{\dot{W}}_s + Y_s' (\underline{\dot{\mathbb{W}}} + W_{s,r} \underline{\dot{W}})$$

$$= \underbrace{(Y_s + Y_s' W_{s,r})}_{Y_r + O(|s-r|^{2\alpha})} \underline{\dot{W}}_s + \underbrace{Y_s'}_{Y_r' + O(|s-r|^\alpha)} \underline{\dot{\mathbb{W}}}.$$

so $\underline{\dot{Z}}_s$ is a modeled distn of regularity $3\alpha - 1 > 0$.

Proof of the reconstruction theorem. (follow the proof in [FH, §13.4])

(original proof in [Hai13] used Daubechies wavelets).

Theorem Let $M = (\Pi, \Gamma)$ be a model for a regularity structure $\mathcal{T}$ on $\mathbb{R}^d$.

Suppose $\gamma > 0$.

Then there is a unique linear map $R = R_M : \mathcal{D}^\gamma_M \longrightarrow \mathcal{D}'(\mathbb{R}^d)$ s.t.

$$|\langle R f - \Pi_x(f(x)), \varphi_x^\lambda \rangle| \lesssim \lambda^\gamma.$$

To start, we build a special mollifier. Let $\alpha > 0$, and let $\rho : \mathbb{R}^d \to \mathbb{R}$ be even, smooth, cptly-supported in a ball of radius 1, with

$$\int x^k \rho(x)\,dx = \delta_{k,0} \qquad 0 \leq |k| \leq \alpha.$$

(k is a multi-index)

$\boxed{\text{DO:}}$ check that such a ρ exists.

Define $\rho^{(n)}(x) = 2^{nd} \rho(2^n x)$

& $\rho^{(n,m)} = \rho^{(n)} * \rho^{(n+1)} * \cdots * \rho^{(m)}$.

Let $\varphi^{(n)} = \lim_{m \to \infty} \rho^{(n,m)}$ $\boxed{\text{DO:}}$ check that the limit exists.

$\Downarrow$

use Fourier.

Then $\varphi^{(n)} = \rho^{(n)} * \varphi^{(n+1)}$.

& define $\rho_x^{(n)}(y) = \rho^{(n)}(y - x)$.

Lemma. Let $\alpha>0$, and let $\xi_n : \mathbb{R}^d \to \mathbb{R}$ be functions s.t.

$\forall$ cpt K, $\sup_{\substack{x\in K\\ n\in\mathbb{N}}} \frac{|\xi_n(x)|}{2^{\alpha n}} < \infty$

and $\xi_n = \rho^{(n)} * \xi_{n+1}$. Then the sequence (ξ_n) is Cauchy in the topology of $C^{-\beta}$ for each $\beta>\alpha$. Moreover, the limit ξ satisfies $\xi_n = \varphi^{(n)} * \xi$.

Moreover, if for some $x\in\mathbb{R}^d$ and $\gamma>-\alpha$, we have $|\xi_n(y)| \le 2^{\alpha n}(|x-y|^{\gamma+\alpha} + 2^{-(\gamma+\alpha)n})$ uniformly over $n\geq 0$ and $|x-y|\le 1$, then

$$|\langle \xi, \psi_x^\lambda \rangle| \lesssim \lambda^\gamma \text{ for } \lambda\le 1.$$

Proof. Let $\lambda\in(0,1]$, let ψ_λ be a test fn supported in a ball of radius λ s.t. $|D^k\psi| \lesssim \lambda^{-d-|k|}$ for $|k|\le\alpha+1$.

To show that (ξ_n) is Cauchy in $C^{-\beta}$, suffices to bound

$$|\psi_\lambda * (\xi_n - \xi_{n+1})| \lesssim \lambda^{-\beta} 2^{(\alpha-\beta)n}$$

locally uniformly in x, w/ prop. const. bounded independently of ψ_λ.

Consider two cases:

first, if $\lambda\le 2^{-n}$, then just use triangle inequality and the bound on $|\xi_n(x)|$.

So only need to consider the case $\lambda\geq 2^{-n}$.

Rewrite the lhs:

$$\psi_\lambda * (\xi_n - \xi_{n+1}) = (\psi^\lambda * \rho^{(n)} - \psi^\lambda) * \xi_{n+1}.$$

Then use Taylor expansion to write

$$|\psi^\lambda(y) - T_x^{(\alpha)}(y)| \lesssim \lambda^{-N-d} |y-x|^N \quad (*)$$

where $N = \lceil \alpha \rceil$ and

$$T_x^{(\alpha)}(y) := \sum_{|k| \le \alpha} \frac{D^k \psi^\lambda(x)}{k!} (y-x)^k.$$

$\boxed{\text{Do!}}$ $\rho^{(n)} * T_x^{(\alpha)} = T_x^{(\alpha)}$. Also, $T_x^{(\alpha)}(x) = \psi^\lambda(x)$.

So: $(\psi^\lambda * \rho^{(n)} - \psi^\lambda)(x) = (\rho^{(n)} * \underbrace{(\psi_\lambda - T_x^{(\alpha)})}_{\text{controlled by } (*)})(x)$.

So in fact $|\psi^\lambda * \rho^{(n)} - \psi^\lambda| \lesssim \lambda^{-N-d} 2^{-nN}$.

also, support has diameter 2λ, so $\|\psi^\lambda * \rho^{(n)} - \psi^{(\lambda)}\|_{L^1} \le \lambda^{-N} 2^{-nN}$.

On the other hand,

$|\xi_{n+1}| \lesssim 2^{\alpha n}$, so

$$|\psi^\lambda * (\xi_n - \xi_{n+1})| = |(\psi^\lambda * \rho^{(n)} - \psi^\lambda) * \xi_{n+1}| \le \lambda^{-N} 2^{(\alpha - N)n}.$$

This proves the desired result for $\beta \in [\alpha, N]$. For larger β, choose α larger.

For the second claim:

[assume $|\xi_n(y)| \le 2^{\alpha n}\left(|x-y|^{\gamma+\alpha} + 2^{-(\gamma+\alpha)n}\right)$]

$$\langle \xi, \psi_x^\lambda \rangle = \langle \xi_n, \psi_x^\lambda \rangle + \sum_{k \ge n} \langle \xi_{k+1} - \xi_k, \psi_x^\lambda \rangle,$$

where we choose n s.t. $\lambda \in [2^{-(n+1)}, 2^{-n}]$.

For the first term, bound

$$|\langle \xi, \psi_x^\lambda \rangle| \lesssim \lambda^{-d} \int_{B_x(\lambda)} 2^{\alpha n}\left(|x-y|^{\gamma+\alpha} + 2^{-(\gamma+\alpha)n}\right) dy$$

$$\lesssim \lambda^{\gamma+\alpha} 2^{\alpha n} + 2^{-\gamma n}$$

$$\lesssim \lambda^{\gamma}.$$

For the second terms (the telescoping sum), write using the Taylor expansion trick

$$\langle \xi_{k+1} - \xi_k, \psi_x^\lambda \rangle$$

$$\lesssim \lambda^{-N-d} 2^{-nN} \int_{B_x(2\lambda)} |\xi_{n+1}(y)| dy$$

$$\lesssim \lambda^{\gamma+\alpha-N} 2^{(\alpha-N)n} + \lambda^{-N} 2^{-(\gamma+N)n}.$$

This is summable since $N > \alpha$ and $N > -\gamma$ and the sum is of order λ^γ.

Next: Let's use this to prove the reconstruction theorem.

Proof of reconstruction theorem:

$$\left(R^{(m,m)} f\right)(y) = \left(\varphi^{(m)} * \Pi_y f(y)\right)(y)$$

goal: construct $\lim_{m\to\infty} R^{(m,m)} = R$.

For $m > n$, define

$$R^{(n,m)} f = \rho^{(n,m-1)} * R^{(m,m)} f.$$

So $\left(R^{(n,m)} f - R^{(n,m+1)} f\right)(x)$

$$= \int dy\, \rho_x^{(n,m-1)}(y) \int dz\, \rho_x^{(m)}(y) \Big\langle \Pi_y \underbrace{\big(f(y) - \Gamma_{yz} f(z)\big)}_{\text{component in } T_\alpha \text{ is bounded by } 2^{(\alpha-\gamma)m}} \Big), \varphi_z^{(m+1)} \Big\rangle$$

$\Pi_y \Gamma_{yz} = \Pi_z$

So $\lesssim \sum_\alpha 2^{-\alpha m} \cdot 2^{(\alpha-\gamma)m} \lesssim 2^{-\gamma m}$.

so in fact $\left\| \left(R^{(n,m)} - R^{(n,m+1)}\right) f \right\|_{L^\infty} \lesssim 2^{-\gamma m}$.

so $R^{(n)} f = \lim_{m\to\infty} R^{(n,m)} f$ exists.

it also satisfies

$$\|R^{(n)} f\|_{L^\infty} \lesssim 2^{-\alpha_0 n}$$

$\min A$.

since this is easy to check for $R^{(n,n)}$

Moreover, $R^{(n)} f = \rho^{(n)} * R^{(n+1)} f$.

So: apply the lemma to get

$R f = \lim_{n\to\infty} R^{(n)} f$ exists in C^α $\forall\, \alpha < \alpha_0$.

outline of the rest of the course:

1. operations on regularity structures (differentiate, multiply, apply heat kernels)

2. how to set up a fixed point argument to solve KPZ.