Reconstruction theorem - last part. (8/20)

goal: $f \in \mathcal{D}^\gamma$

$$|\langle Rf - \Pi_x(f(x)), \varphi_x^\lambda \rangle| \lesssim \lambda^\gamma.$$

$$\Leftarrow \|f(x) - \Gamma_{xy} f(y)\|_\beta \lesssim |x-y|^{\gamma-\beta}.$$

Proof. Define $f_x(y) = \Gamma_{yx} f(x)$.

Then

Do: $R^{(n)} f_x = \varphi^{(n)} * \Pi_x f(x)$.

so taking the limit $n \to \infty$ we see that

$$R f_x = \Pi_x(f(x)).$$

so goal becomes $|\langle R(f - f_x), \varphi_x^\lambda \rangle| \lesssim \lambda^\gamma$.

In fact,

$$R^{(n,n)}(f - f_x)(y)$$

$$= \langle \Pi_y (f - f_x), \varphi_y^{(n)} \rangle$$

$$\lesssim \sum_{\alpha < \gamma} 2^{-\alpha n} |y-x|^{\gamma - \alpha}$$

because $\|(f - f_x)(y)\|_\alpha \lesssim |y-x|^{\gamma-\alpha}$.

$$\lesssim 2^{-\alpha_0 n}\left(|y-x|^{\gamma-\alpha_0} + 2^{(\alpha_0-\gamma)n}\right)$$

where α_0 is the minimum degree in A.

Then the second part of the lemma we proved last time shows the conclusion.

Recall: $\mathcal{R}^{(n,m)} f(y) = \left(\varphi^{(m)} * \Pi_y \mathbf{f}(y)\right)(y)$

$$\mathcal{R}^{(n,m)} f(y) = \left(\rho^{(n,m-1)} * \mathcal{R}^{(m,m)} \mathbf{f}\right)(y)$$

$$\mathcal{R}^{(n)} \mathbf{f} = \lim_{m \to \infty} \mathcal{R}^{(n,m)} \mathbf{f}.$$

$$\mathcal{R} \mathbf{f} = \lim_{n \to \infty} \mathcal{R}^{(n)} \mathbf{f}.$$

$f \in C^\gamma \qquad \gamma \in (2,3)$

$$\mathbf{f}(x) = f(x)\,\mathbf{1} + f'(x)\,\mathbf{X} + \frac{1}{2} f''(x)\,\mathbf{X}^2.$$

$$\mathcal{R} \mathbf{f}(x) = f(x)$$

Reconstruction thm:

$$\underset{\substack{\shortparallel \\ f(y)}}{\mathcal{R} \mathbf{f}(y)} - \Pi[\mathbf{f}(x)] = f(y) - \left[f(x) + f'(x)(y-x) + \frac{1}{2} f''(x)(y-x)^2\right]$$

$$= O(|y-x|^\gamma).$$

Operations on regularity structures

Motivation: KPZ eqn

$$\underbrace{\partial_t h - \tfrac{1}{2}\Delta h}_{\text{linear part}} = \underbrace{(\partial_x h)^2}_{\text{nonlinearity}} + \xi.$$

need to represent:

- differentiation.
- multiplication (squaring)
- convolution w/ heat kernel

Differentiation. Recall a sector of a regularity structure is a subspace $V \subseteq T$ s.t. V is itself a RS.

Defn If $V \subseteq T$ is a sector, then a lin. op. $\underline{\partial}: V \to T$ realizes ∂_x for a model (Π, Γ) if

- $\underline{\partial}\tau \in T_{\alpha-1}$ $\forall \tau \in T_\alpha \cap V$
- $\Gamma \underline{\partial}\tau = \underline{\partial}\Gamma\tau$ $\forall \tau \in V$, $\Gamma \in G$.
- $\Pi_x(\underline{\partial}\tau)(y) = \frac{\partial}{\partial y}(\Pi_x \tau)(y)$ $\forall \tau \in V$

DO: check for polynomials $\underline{\partial}\underline{X}^k = k\underline{X}^{k-1}$ satisfies this

Prop. Let $f \in \mathcal{D}^\gamma(V)$ ($\gamma > 1$). Then $\underline{\partial} f \in \mathcal{D}^{\gamma-1}$ and $\boxed{\mathcal{R}\underline{\partial} f = \partial_x \mathcal{R} f}$

Part of the proof: Check second claim.

goal: $\forall$ test fns ψ, $x \in \mathbb{R}^d$, we have

$$|\langle \Pi_x \underline{\partial} f(x) - \partial_x \mathcal{R} f, \psi_x^\lambda \rangle| \lesssim \lambda^\delta.$$

for some $\delta > 0$.

To see this, write

$$\langle \underbrace{\Pi_x \partial f(x)}_{\partial_x \Pi_x f(x)} - \partial_x \mathcal{R} f, \varphi_x^\lambda \rangle$$

$$= - \langle \Pi_x f(x) - \mathcal{R} f, \partial_x \psi_x^\lambda \rangle |$$

$$\lesssim \lambda^{\gamma - 1}!$$

So proof is complete w/ $\delta = \gamma - 1 > 0$.

<u>Multiplication</u>. Given <u>two</u> sectors V_1, V_2, a <u>product</u> $\star$ is a bilinear map

$\star : V_1 \times V_2 \to T$ s.t. if $\tau_1 \in V_1 \cap T_\alpha$, $\tau_2 \in V_2 \cap T_\beta$

then $\tau_1 \star \tau_2 \in T_{\alpha+\beta}$

and for any $\Gamma \in G$, $\Gamma(\tau_1 \star \tau_2) = \Gamma\tau_1 \star \Gamma\tau_2$

<u>DO</u>: for poly RS, the usual product is a product in this sense.

Let $\mathcal{D}^\gamma_\alpha = \{ f \in \mathcal{D}^\gamma : f(x) \in \bigoplus_{\beta \geq \alpha} T_\beta \ \forall x \in \mathbb{R}^d \}$

<u>Theorem</u>. If $f_1 \in \mathcal{D}^{\gamma_1}_{\alpha_1}$, $f_2 \in \mathcal{D}^{\gamma_2}_{\alpha_2}$

then $f_1 \star f_2 \in \mathcal{D}^\gamma_\alpha$

with $\alpha = \alpha_1 + \alpha_2$

$\gamma = (\gamma_1 + \alpha_2) \wedge (\gamma_2 + \alpha_1)$

Examples.

$$\cancel{\left(f\,\underline{1} + f'\,\underline{X} + \tfrac{1}{2}f''\,\underline{X}^2\right)\left(g\,\underline{1} + g'\,\underline{X} + \tfrac{1}{2}g''\,\underline{X}^2\right)}$$

$$\cancel{= (fg)\,\underline{1} + \cdots + \tfrac{1}{2}(fg'')\,\underline{X}^2.}$$

$$\left(f_1\,\overset{\alpha<0}{\underline{\Xi}_1} + f_2\,\overset{\beta>0}{\underline{\Xi}_2}\right) \star \left(g_1\,\overset{\alpha<0}{\underline{\Xi}_1} + g_2\,\overset{\beta>0}{\underline{\Xi}_2}\right)$$

$$(f_1 g_1)(\underline{\Xi}_1 \star \underline{\Xi}_2) + \underbrace{(f_1 g_2)}_{\text{has regularity of } g_2.}\underbrace{(\underline{\Xi}_1 \star \underline{\Xi}_2)}_{\alpha+\beta<\beta.}$$

Integration against the heat kernel.

<u>goal</u>: if we have a kernel K that improves regularity by β ($=2$ for heat kernel)

we want to lift it to an operator
$$\mathcal{K}: \mathcal{D}^{\gamma} \to \mathcal{D}^{\gamma+\beta}$$

s.t. $$\mathcal{R}\mathcal{K} f = K * \mathcal{R} f.$$

for heat kernel, space-time convolution

<u>First step</u>: introduce some operator $\tilde{I}$ on T (really $\tilde{I} : V \to T$ for some sector V)

s.t. $\tilde{I} V_\alpha \subseteq T_{\alpha+\beta}$.

This seems kind of bizarre because $\tilde{I}\,\mathbf{1} = ?$

maybe it should be spatially constant; so maybe involves something times $\mathbf{1}$..?

again see something of degree 0. !

so to make $\tilde{I}$ have the desired degree cond, it does not <u>by itself</u> represent convolution, but rather convolution and then subtracting off the regular pie