Schauder estimates / heat kernel.

given some kernel K that "improves regularity by $\beta=2$", we want to construct a lift $\mathcal{K}: \mathcal{D}^\gamma \to \mathcal{D}^{\gamma+\beta}$ such that

$$\mathcal{R}\mathcal{K} f = K * \mathcal{R} f$$

<u>Assumption 1</u>. The regularity structure contains a sector $T_{poly} \subseteq T$ such that T_{poly} is isomorphic to the polynomial RS.

<u>Assumption 2</u>. There is a sector $V \subseteq T$ and a linear map $\mathcal{I}: V \to T$ such that $\mathcal{I}(V \cap T_\alpha) \subseteq T_{\alpha+\beta}$ and for every $\Gamma \in G$, $\tau \in T$, $\Gamma \mathcal{I} \tau - \mathcal{I} \Gamma \tau \in T_{poly}$.

How should this play with the model?

Given a β-regularizing kernel K and a RS T satisfying the last two assumptions, we say that a model (Π, Γ) is <u>admissible</u> if

$$(\Pi_x X^k)(y) = (y-x)^k.$$

and

$$\Pi_x(\mathcal{I}\tau) = K * \Pi_x \tau - \Pi_x \mathcal{J}_x \tau$$

where for $\tau \in T_\alpha$,

$$\mathcal{J}_x \tau = \sum_{|k| < \alpha+\beta} \frac{X^k}{k!} \int (D^k K)(x-y) \cdot (\Pi_x \tau)(dy)$$

typical ex: $\alpha = -3/2$, $\beta = 2$.
then $\mathcal{J}_x \tau = \mathbf{1} \cdot (K * (\Pi_x \tau))(x)$.

Given this setup, we can define the operator $\mathcal{K}: \mathcal{D}^{\gamma} \to \mathcal{D}^{\gamma+\beta}$.

$$(\mathcal{K} f)(x) = \mathcal{I}(f(x)) + \mathcal{J}_x(f(x)) + (\mathcal{N} f)(x)$$

where

$$(\mathcal{N} f)(x) = \sum_{|k|<\gamma+\beta} \frac{X^k}{k!} \int D^k K(x-y)\big(\mathcal{R} f - \Pi_x f(x)\big)(dy)$$

Theorem. Let K be a β-regularizing kernel, $\mathcal{T}$ be a RS satisfying the assumptions, and let (Π, Γ) be an admissible model. Then for every $f \in \mathcal{D}^{\gamma}$ (w/ some technical assumptions on γ e.g. $\gamma+\beta \notin \mathbb{N}$), we have

$$\mathcal{K} f \in \mathcal{D}^{\gamma+\beta}, \text{ and } \mathcal{R}\mathcal{K} f = K * \mathcal{R} f.$$

↳ "multilevel Schauder estimate".

Let's give a proof of the last part in the case where the model consists of continuous functions,
so $\mathcal{R} f(x) = (\Pi_x f(x))(x)$.

In that case,

$$(\mathcal{R}\mathcal{K} f)(x) = \Pi_x\big(\mathcal{I} f(x) + \mathcal{J}_x f(x)\big)(x) + \big(\Pi_x (\mathcal{N} f)(x)\big)(x)$$

For the first term,

$$\Pi_x\big(\mathcal{I} f(x) + \mathcal{J}_x f(x)\big)(x) = K * \Pi_x f(x).$$

cancel

For the second term, only $k=0$ contributes, so

$$\big(\Pi_x (\mathcal{N} f)(x)\big)(x) = \int K(x-y)\big(\mathcal{R} f - \Pi_x f(x)\big)(dy)$$

so $(\mathcal{R}\mathcal{K} f)(x) = (K * \mathcal{R} f)(x)$.

KPZ Equation

How can we use this to set up solutions to the KPZ equation?

$$\boxed{\partial_t h - \frac{1}{2}\Delta h = \frac{1}{2}(\partial_x h)^2 + \xi}$$

First remark: we need to use "parabolic rescaling"

$$\varphi^\lambda_{(s,x)}(t,y) = \lambda^{-3}\varphi(\lambda^{-2}(t-s), \lambda^{-1}(y-x)).$$

and when we count the degree of a polynomial, time counts as degree 2.

e.g. $\deg(\underline{T}^4 \underline{X}^3) = 2\cdot 4 + 1\cdot 3 = 11.$

w/ this way of measuring, one can check that w.p.1, $\xi \in C^\alpha$ for every $\alpha < -\frac{3}{2}$.

since $(E\langle \xi, \varphi^\lambda_x\rangle^2)^{1/2} = \|\varphi^\lambda_x\|_{L^2} \sim \lambda^{-3/2}$.

Mild solution formula:

$$h = \underbrace{G * h_0}_{\text{spatial conv.}} + G \circledast \left[\frac{1}{2}(\partial_x h)^2 + \xi\right].$$

goal: Lift this to an equation of modeled distributions of the form

$$\underline{H} = \underbrace{G_t * h_0}_{\text{polynomial lift}} + \overset{\text{lift of } G.}{\mathcal{G}}\left[\frac{1}{2}(\partial \underline{H})^{\star 2} + \underset{\text{lifted noise}}{\underline{\Xi}}\right]$$

First question: what terms should we include in the RS besides polynomials?

RHS: $G\left[\frac{1}{2}(\partial \underline{H})^2 + \underline{\Xi}\right]$.

$\underline{H}$ = poly + ...

Fix $\kappa > 0$ very small. irrational.

First, we should add $\underline{\Xi} = \bullet$

Then add $\mathcal{I}\tau$ = . Add 2 to deg($\bullet$)

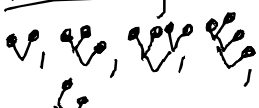
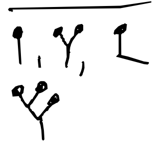
Next, $\partial \mathcal{I}\tau = \mathcal{I}'\tau$ = .

Multiply ⋆ =: . Then get ,

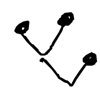
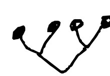
. Also have $\mathcal{I}(\underline{1}\cdot$ $)$ = , $\mathcal{I}'(\underline{1}\cdot$ $)$ =

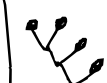
...

why can we stop here? It suffices to truncate the regularity structure at a degree γ such that our PDE is well-posed in $\mathcal{D}^\gamma$. Let's figure out what that γ should be.

noise	nonlinearity	solution	solution'
			

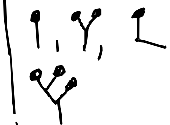
Let's construct solution' in $\mathcal{D}^{1/2+10K}_{-1/2-K}$

τ		$\deg(\tau)$
		$-3/2-K$.
		$1/2-K$.
	lollipop	$-1/2-K$.
	cherry	$-1-2K$.
		$1-2K$
		$-2K$
		$3/2-K$
		$1/2-K$.
	elk	$-1/2-3K$
	claw	$-2K$
	candelabra	$-4K$
		$3/2-3K$
		$1/2-3K$

τ	$\deg \tau$
 moose	$-4K$

$\underline{H} = \text{poly} + \mathcal{G}[\tfrac{1}{2}(\partial \underline{H})^2 + \underline{\xi}]$, $\underline{U} = \partial \underline{H}$

$\underline{U} = \text{poly} + \partial\mathcal{G}[\tfrac{1}{2}\underline{U}^2 + \underline{\xi}]$

noise	nonlinearity	solution	solution'
			

e.g. deg() $= 1-4K$

let's construct solution' in $\mathcal{D}^{1/2+10K}_{-1/2-K}$

☆ : $\mathcal{D}^{1/2+10K}_{-1/2-K} \times \mathcal{D}^{1/2+10K}_{-1/2-K} \longrightarrow \mathcal{D}^{9K}_{-1-2K}$

so construct the nonlinearity in $\mathcal{D}^{9K}_{-1-2K}$

Then when we integrate and then differentiate, we get $\partial\mathcal{G}[(\partial \underline{H})^2] \in \mathcal{D}^{1+9K}_{-2K}$.

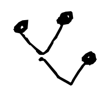
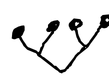
OTOH: $\partial\mathcal{G}\,\underline{\xi} \in \mathcal{D}^{\infty}_{-1/2-K}$.

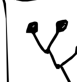
so altogether get solution' $\in \mathcal{D}^{1+9K}_{-1/2-K}$

$\subseteq \mathcal{D}^{1/2+10K}_{-1/2-K}$.

two main points: ① can close the argument.

② $9K>0$ so can reconstruct the forcing = noise + nonlinearity

τ		$\deg(\tau)$
		$-3/2-K$
		$1/2-K$
	lollipop	$-1/2-K$
	cherry	$-1-2K$
		$1-2K$
		$-2K$
		$3/2-K$
		$1/2-K$
	elk	$-1/2-3K$
	claw	$-2K$
	candelabra	$-4K$
		$3/2-3K$
		$1/2-3K$

τ	$\deg\tau$
 moose	$-4K$

Next time: how to build the model
(& the structure group).
(at that point we will need
to renormalize)