Minicourse: Semilinear stochastic heat equations in the critical dimension

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1 Motivation, examples, and phenomenology

This minicourse will concern the analysis of semilinear stochastic heat equations, which is to say equations taking the form

$$\mathrm{d}u_t(x) = \frac{1}{2}\Delta u_t(x)\mathrm{d}t + \sigma(u_t(x))\mathrm{d}W_t(x). \tag{1}$$

Here, $t \in \mathbb{R}$, $x \in \mathbb{R}^d$ or $(\mathbb{R}/\mathbb{Z})^d$ for some $d \in \mathbb{N}$, $\sigma\colon \mathbb{R} \to \mathbb{R}$ is a nonlinearity, and $\mathrm{d}W_t(x)$ is a(n approximation of a) Gaussian space-time white noise, i.e. $\mathrm{d}W_t(x)$ is mean-zero Gaussian and we have

$$\mathbb{E}\left[\mathrm{d}W_t(x)\mathrm{d}W_{t'}(x')\right] = \delta(t-t')R(x-x'),$$

with $R(x) \approx \delta(x)$. It turns out that when $d > 1$, it is often difficult to make sense of the equation with $R(x) = \delta(x)$, and so we consider a regularization. More on this later!

1.1 Examples

Semilinear stochastic heat equations often arise in the modeling of stochastic systems. Here are some popular examples of choices of σ that can arise in applications:

Edwards–Wilkinson When $\sigma \equiv 1$, we have the *additive stochastic heat equation* or *Edwards–Wilkinson equation*. This equation has Gaussian solutions so the analysis is in some sense trivial. However, it arises as the scaling limit of many stochastic systems in which the interaction is weak enough that the limit doesn't see any interaction beyond the Laplacian and the Gaussian noise. We will discuss the Edwards–Wilkinson equation later in this lecture; the book [9] has a lot of interesting background.

Multiplicative stochastic heat equation and directed polymers When $\sigma(u) = \beta u$, we have the *multiplicative stochastic heat equation*. This equation is linear in the initial condition, but not in the noise. Because of this, if R is spatially smooth, it can be solved via the Feynman–Kac formula:

$$\begin{aligned} u_t(x) &= \mathrm{E}^{t,x} \exp\left\{\beta\int_0^t \mathrm{d}W_s(X_s)\,\mathrm{d}s - \frac{1}{2}\beta^2\int_0^t \mathrm{d}[W(0)]_s\right\} \\ &= \mathrm{E}^{t,x} \exp\left\{\beta\int_0^t \mathrm{d}W_s(X_s)\,\mathrm{d}s - \frac{1}{2}\beta^2 R(0)t\right\}. \end{aligned} \tag{2}$$

(See [2], and also [1] for the $R = \delta$ case.) Here, under the measure $\mathrm{E}^{t,x}$ on $\mathcal{C}([0,t])$, the process $(X_s)_{s\in[0,t]}$ is a backward-in-time Brownian motion with $X_t = x$. We note that we use lightface E for the expectation over the Brownian motion, as opposed to the boldface $\mathbb{E}$ for expectation over the noise.

The Feynman–Kac formula (2) is the key to the relationship between the multiplicative stochastic heat equation and *directed polymer* models. In a directed polymer model, we introduce a random measure $\hat{\mathrm{P}}^{t,x}$ on $\mathcal{C}([0,t])$ that is absolutely continuous with respect to $\mathrm{P}^{t,x}$ with Radon–Nikodym derivative

$$\frac{\mathrm{d}\hat{\mathrm{P}}^{t,x}}{\mathrm{d}\mathrm{P}^{t,x}}(Y) = \frac{\exp\left\{\beta\int_0^t \xi_s(Y_s)\,\mathrm{d}s - \frac{1}{2}\beta^2 R(0)t\right\}}{\mathrm{E}^{t,x}\exp\left\{\beta\int_0^t \xi_s(X_s)\,\mathrm{d}s - \frac{1}{2}\beta^2 R(0)t\right\}}. \tag{3}$$

The denominator in (3) is present to ensure that

$$\mathrm{E}^{t,x}\left[\frac{\mathrm{d}\hat{\mathrm{P}}^{t,x}}{\mathrm{d}\mathrm{P}^{t,x}}(X)\right] = 1 \qquad \text{a.s.},$$

so $\hat{\mathrm{P}}^{t,x}$ is almost surely a probability measure. Under the measure $\hat{\mathrm{P}}^{t,x}$, the path X tends to favor regions of space-time in which the noise takes larger values. The directed polymer model has been studied extensively; some entry points to the literature are [5, 14].

Super-Brownian motion When $\sigma(u) = \sqrt{u}$, the *super-Brownian motion* process solves the SPDE (1) in a certain sense (that of the *martingale problem*). What this means, since in this case σ is non-Lipschitz, is beyond the scope of these notes; we refer to [8, 11, 13]. This process arises as the scaling limit of the empirical particle density of a branching Brownian motion with appropriate critical scaling. The square-root nonlinearity, roughly speaking, comes about because since each particle branches and dies independently, the variance of the change in the number of particles is proportional to the local number of particles.

Voter model When $\sigma(u) = \sqrt{u(1-u)}$, the SPDE (1) can in some regimes be related to the voter model. This has been done in [10] for a long-range version of the voter model in $d = 1$. Informally speaking, this choice of nonlinearity comes about because, in the voter model, the variance of the change in the number of "yes" voters is proportional to the local number of disagreements, which is $u(1-u)$ if u is the number of "yes" voters.

1.2 Phenomenology of the Edwards–Wilkinson equation

In the case when $\sigma \equiv 1$, the solution u to (1) is Gaussian and, in principle, everything is easy to understand. However, we'll work things out in some detail, because it helps motivate the cases that arise in the study of the nonlinear problems. It turns out that the phenomenology of the solution depends significantly on the spatial dimension d, and the relevant cases are $d = 1$, $d = 2$, and $d \geq 3$.

Let

$$G_t(x) = \frac{1}{(2\pi t)^{d/2}}\exp\left\{-\frac{|x|^2}{2t}\right\}$$

denote the standard heat kernel. By the Duhamel formula, we have

$$u_t(x) = G_t * u_0(x) + \int_0^t G_{t-s} * \mathrm{d}W_s(x).$$

For simplicity, for now let's assume that $u_0 \equiv 0$, so we have

$$u_t(x) = \int_0^t G_{t-s} * \mathrm{d}W_s(x).$$

This is a Gaussian process and we can compute the covariance: for $t' > t$, we have

$$\mathbb{E}[u_t(x)u_{t'}(x')] = \int_0^t \iint G_{t-s}(x-y)R(y-y')G_{t'-s}(y'-x')\,\mathrm{d}y\,\mathrm{d}y'\,\mathrm{d}s = \int_0^t G_{t+t'-2s} * R(x-x')\,\mathrm{d}s.$$

In particular, if $R = \delta$, we get

$$\mathbb{E}\left[u_t(x)^2\right] = \int_0^t G_{2(t-s)}(0)\,\mathrm{d}s = \int_0^t \frac{\mathrm{d}s}{(4\pi(t-s))^{d/2}},$$

which is finite only when $d < 2$. This implies:

Lemma 1.1. *The Edwards–Wilkinson equation has function-valued solutions only when $d = 1$. In higher-dimensions, it has distribution-valued solutions.*

What about at large times? Let

$$Q_t(x) = \mathbb{E}[u_t(x)u_t(0)] = \int_0^t G_{2(t-s)} * R(x)\,\mathrm{d}s = \int_0^t G_{2s} * R(x)\,\mathrm{d}s.$$

We note (somewhat formally) that

$$\Delta Q_t(x) = \int_0^t \Delta G_{2s} * R(x)\,\mathrm{d}s = -\int_0^t \frac{\mathrm{d}}{\mathrm{d}s} G_{2s} * R(x)\,\mathrm{d}s = R(x) - G_{2t} * R(x) \to R(x) \qquad \text{as } t \to \infty.$$

The problem

$$\Delta Q_t(x) = R(x)$$

has bounded solutions if and only if $d > 2$. This suggests the following (which is true):

Lemma 1.2. *If $d > 2$, then the Edwards–Wilkinson equation has invariant measures which are given by R convolved with the law of a Gaussian free field. (If you don't know what the Gaussian free field is, this is a reasonable definition.)*

Thus we have a sort of dichotomy: if $d < 2$, then the Edwards–Wilkinson equation is "nice" locally (has function-valued solutions), while if $d > 2$, then the Edwards–Wilkinson equation is "nice" globally (has invariant measures).

1.3 The Edwards–Wilkinson equation in $d = 2$

In $d = 2$, neither of the above situations is the case: the equation has "problems" both "locally" and "globally." In fact, this makes a lot of sense, because of the following proposition.

Proposition 1.3. *Let $d = 2$ and let $u_t(x)$ solve the Edwards–Wilkinson equation with $u_0 \equiv 0$ and $R = \delta$. Let $\kappa \in (0, \infty)$ be a deterministic constant and let $u^\kappa_s(y) = u_{\kappa^{-2}s}(\kappa^{-1}y)$. Then* $\mathrm{Law}(u^\kappa) = \mathrm{Law}(u)$.

Proof. We just have to compare the covariances. We have

$$\begin{aligned}
\mathbb{E}[u^\kappa_t(x)u^\kappa_{t'}(x')] &= \mathbb{E}[u_{\kappa^{-2}t}(\kappa^{-1}x)u_{\kappa^{-2}t'}(\kappa^{-1}x')] \\
&= \int_0^{\kappa^{-2}t} G_{\kappa^{-2}(t+t')-2s}(\kappa^{-1}[x-x'])\,\mathrm{d}s \\
&= \kappa^{-2}\int_0^t G_{\kappa^{-2}(t+t'-2s)}(\kappa^{-1}[x-x'])\,\mathrm{d}s \\
&= \int_0^t G_{t+t'-2s}(x-x')\,\mathrm{d}s \\
&= \mathbb{E}[u_t(x)u_{t'}(x')],
\end{aligned}$$

where we used the fact that, in $d=2$,

$$\kappa^{-2}G_{\kappa^{-2}t}(\kappa^{-1}x) = \kappa^{-2}\cdot\frac{1}{2\pi\kappa^{-2}t}\exp\left\{-\frac{|\kappa^{-1}x|^2}{2\kappa^{-2}t}\right\} = G_t(x). \qquad \square$$

The last proposition means that the two-dimensional Edwards–Wilkinson equation has a *scale-invariant* character. The major question that we will answer in this minicourse is, "how can we study scale-invariant *semilinear* stochastic heat equations?"

The Edwards–Wilkinson equation with $R=\delta$ has distribution-valued solutions in $d=2$. Hence, if we consider a more general semilinear SHE in $d=2$, we wouldn't expect it to have function-valued solutions either. Since we can't apply a general nonlinearity σ to a distribution, this means that we do not know how to interpret the nonlinear problem (1) when $d=2$ and $R=\delta$.

To rectify this, let's fix a small parameter $\varepsilon>0$ and let $R=R^\varepsilon=\frac{1}{\log\varepsilon^{-1}}G_{\varepsilon^2}$. (Actually, we should replace G_{ε^2} by a general mollifier at scale ε, but this choice simplifies the calculations and doesn't really lose anything.) This means that we are mollifying the noise, so the equation will be well-posed, but also *attenuating* the noise, so it becomes weaker as $\varepsilon\to 0$. If u^ε is the solution to the equation with this choice of ε and $u_0^\varepsilon\equiv 0$ (here we abuse notation since this is not the same definition as u^κ above), then we have

$$\begin{aligned}
\mathbb{E}[u_t(x)^2] &= \frac{1}{\log\varepsilon^{-1}}\int_0^t G_{2(t-s)}*G_{\varepsilon^2}(0)\,\mathrm{d}s = \frac{1}{2\log\varepsilon^{-1}}\int_{2\varepsilon^2}^{t+2\varepsilon^2} G_s(0)\,\mathrm{d}s = \frac{1}{2\log\varepsilon^{-1}}\int_{2\varepsilon^2}^{t+2\varepsilon^2}\frac{1}{2\pi s}\,\mathrm{d}s \\
&= \frac{1}{4\pi}\log_{\varepsilon^{-1}}\frac{t+2\varepsilon^2}{2\varepsilon^2}. \qquad (4)
\end{aligned}$$

If t is fixed and $\varepsilon\to 0$, then the right side converges to $\frac{1}{2\pi}$, i.e. an order-1 variance. So with this setup, we get *pointwise* convergence of the random fields as $\varepsilon\to 0$. This is a nice scaling regime because it allows us to consider the same problem with a nonlinearity.

1.3.1 Fluctuation scales

We note that, in this scaling regime, for $x\neq x'$, we have

$$\mathbb{E}[u^\varepsilon_t(x)u^\varepsilon_t(x')] = \frac{1}{2\log\varepsilon^{-1}}\int_{2\varepsilon^2}^{t+2\varepsilon^2} G_s(x-x')\,\mathrm{d}s \to 0 \qquad \text{as } \varepsilon\to 0.$$

That is, distinct points are completely decorrelating as $\varepsilon\to 0$, and the resulting random field is trivial. To see nontrivial correlations between points, we should look, for $\alpha\in(0,1]$ and fixed $y\neq 0$, at

$$\begin{aligned}\mathbb{E}[u_t^\varepsilon(x)u_t^\varepsilon(x+\varepsilon^\alpha y)] &= \frac{1}{2\log\varepsilon^{-1}}\int_{2\varepsilon^2}^{t+2\varepsilon^2} G_s(\varepsilon^\alpha y)\,\mathrm{d}s = \frac{\varepsilon^{2\alpha}}{2\log\varepsilon^{-1}}\int_{2\varepsilon^2}^{t+2\varepsilon^2} G_{\varepsilon^{2\alpha}s}(y)\,\mathrm{d}s \\ &= \frac{1}{2\log\varepsilon^{-1}}\int_{2\varepsilon^{2(1-\alpha)}}^{\varepsilon^{-2\alpha}t+2\varepsilon^{2(1-\alpha)}} G_s(y)\,\mathrm{d}s \sim \frac{1}{2\log\varepsilon^{-1}}\int_1^{\varepsilon^{-2\alpha}t}\frac{1}{2\pi s}\,\mathrm{d}s \to \frac{\alpha}{2\pi} \qquad (5)\end{aligned}$$

as $\varepsilon \to 0$. The "$\sim$" holds because the contribution of $s \lesssim |y|^2 \sim 1$ is negligible.

A special case is when $\alpha = 1$, i.e., we look at two points that are separated by a distance of order ε. In this case, (4) and (5) yield

$$\mathbb{E}\left[(u_t^\varepsilon(x) - u_t^\varepsilon(x+\varepsilon y))^2\right] = 2\mathbb{E}\left[u_t^\varepsilon(x)^2\right] - 2\mathbb{E}\left[u_t^\varepsilon(x)u_t^\varepsilon(x+\varepsilon y)\right] \to 0 \qquad \text{as } \varepsilon \to 0. \qquad (6)$$

Altogether, this means that the solutions to the Edwards–Wilkinson equation with this scaling are *microscopically constant* in the sense that the fluctuations on the "microscopic scale" ε are actually going to zero. To see nontrivial fluctuations, we have to compare two points at a "mesoscopic scale" ε^α.

2 Semilinear stochastic heat equations in $d = 2$

Having explored the behavior of the critical-dimension Edwards–Wilkinson equation, let's now try to add a nonlinearity. It turns out that a similar story to the Edwards–Wilkinson case can be obtained by considering the semilinear equation

$$\mathrm{d}u_t^\varepsilon(x) = \frac{1}{2}\Delta u_t^\varepsilon(x)\mathrm{d}t + \frac{1}{\log\varepsilon^{-1}}\sigma(u_t^\varepsilon(x))\mathrm{d}W_t^\varepsilon(x), \qquad t > 0, x \in \mathbb{R}^2 \qquad (7)$$

and taking the limit as $\varepsilon \to 0$. Here we use the notation $(\mathrm{d}W_t^\varepsilon(x))_{t,x}$ for a Gaussian noise that is white in time and has spatial covariance kernel G_{ε^2}, as above. Our first goal is to show that, for fixed $t > 0$ and $x \in \mathbb{R}^2$, the random variable $u_t^\varepsilon(x)$ converges in law as $\varepsilon \to 0$ to a nontrivial limiting random variable. In fact, we will be able to give a strong characterization of this random variable, depending on the nonlinearity σ.

A little bit of background: this problem was first studied in the linear case $\sigma(u) = \beta u$ by Bertini and Cancrini [3], and the pointwise statistics were studied in this case by Caravenna, Sun, and Zygouras [4]. They proved the following theorem.

Theorem 2.1 ([4]). *Suppose that $\sigma(u) = \beta u$ for some $\beta \in (0, \sqrt{2\pi})$, and $u_0^\varepsilon(x) \equiv 1$. Then, for any fixed $t > 0$, $x \in \mathbb{R}^2$, we have*

$$u_t^\varepsilon(x) \xrightarrow[\varepsilon\downarrow 0]{\text{law}} \exp\left\{Z_\beta - \frac{1}{2}\operatorname{Var}(Z_\beta)\right\} \qquad \text{as } \varepsilon \downarrow 0, \qquad (8)$$

where

$$Z_\beta \sim N\left(0, (1-\beta^2/(2\pi))^{-1}\right). \qquad (9)$$

On the other hand, in the same setting with $\beta \geq \sqrt{2\pi}$, we have

$$u_t^\varepsilon(x) \xrightarrow[\varepsilon\downarrow 0]{\text{law}} 0 \qquad \text{as } \varepsilon \downarrow 0.$$

The methods of [4] are quite interesting and are based on the *polynomial chaos expansion*. The details of this are beyond the scope of this course, but essentially it involves expanding the equation

$$\mathrm{d}u_t^\varepsilon(x) = \frac{1}{2}\Delta u_t^\varepsilon(x)\mathrm{d}t + \frac{1}{(\log\varepsilon^{-1})^{1/2}} u_t^\varepsilon(x)\mathrm{d}W_t^\varepsilon(x)$$

in powers of the noise. Specifically, one writes

$$u_t^{\varepsilon;0}(x) = 1$$

and then

$$\mathrm{d}u_t^{\varepsilon;k}(x) = \frac{1}{2}\Delta u_t^{\varepsilon;k}(x)\mathrm{d}t + \frac{1}{\log\varepsilon^{-1}} u_t^{\varepsilon;k-1}(x)\mathrm{d}W_t^\varepsilon(x)$$

for each $k \geq 1$, and then takes the limit as $k \to \infty$. The point is that each one of the $u^{\varepsilon;k}$s lives in a finite Wiener chaos. It turns out that this expansion yields a series that can be re-summed in a useful way to see the log-normal limit (8) and the limiting variance (7).

In the nonlinear setting, these tools are not available, and in fact we do not expect a log-normal random variable in the limit either. (Indeed, that cannot hold for a general nonlinearity, since we have already seen that for the Edwards–Wilkinson equation we get Gaussian limits). However, the fact that there is a phase transition at $\beta = \sqrt{2\pi}$ is illuminating. Thus, for a while, we will assume that σ is Lipschitz and impose the condition

$$\mathrm{Lip}(\sigma) < \sqrt{2\pi}, \tag{10}$$

which will be in force until further notice. All of our methods will be based on the *mild solution/Duhamel formula* for (7), namely

$$u_t^\varepsilon(x) = G_t * u_0^\varepsilon(x) + \frac{1}{(\log\varepsilon^{-1})^{1/2}} \int_0^t G_{t-s} * \sigma(u_s^\varepsilon)\,\mathrm{d}W_s^\varepsilon. \tag{11}$$

2.1 Moment bound

The first step in our approach will be to prove a one-point moment bound on the solution, which will play a critical role in almost every step of the argument.

Proposition 2.2. *For each $T \in (0,\infty)$, there are constants $\ell > 2$ and $C < \infty$, depending only on σ, T, and u_0^ε, such that*

$$\sup_{t\in[0,T],x\in\mathbb{R}^2} \mathbb{E}[|u_t^\varepsilon(x)|^\ell] \leq C.$$

Proof. We will prove this for $\ell = 2$, which is much simpler than for $\ell > 2$. We will also assume that $\sigma(0) = 0$. For the proof for some $\ell > 2$ and for general σ, see the proof of [6, Proposition 4.4].

We will use the notation

$$|||f||| \coloneqq \sup_{x\in\mathbb{R}^2} \left(\mathbb{E}|f(x)|^2\right)^{1/2}.$$

From (11) we can write by the Itô isometry that

$$\begin{aligned}\mathbb{E}[u_t^\varepsilon(x)^2] &= \mathbb{E}[(G_t * u_0^\varepsilon(x))^2] \\ &\quad + \frac{1}{\log\varepsilon^{-1}} \int_0^t \iint G_{t-s}(x-y_1)G_{t-s}(x-y_2)\mathbb{E}\left[\sigma(u_s^\varepsilon(y_1))\sigma(u_s^\varepsilon(y_2))\right] G_{\varepsilon^2}(y_1-y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}s.\end{aligned} \tag{12}$$

Now we can estimate

$$\mathbb{E}\left[\sigma(u_s^\varepsilon(y_1))\sigma(u_s^\varepsilon(y_2))\right] \le \frac{1}{2}\mathbb{E}\left[\sigma^2(u_s^\varepsilon(y_1))\right] + \frac{1}{2}\mathbb{E}\left[\sigma^2(u_s^\varepsilon(y_1))\right] \le \mathrm{Lip}(\sigma)^2 \|\!|u_s^\varepsilon|\!\|^2$$

and use this in (12) (and also use the fact that y_1 and y_2 play symmetric roles) to get

$$\begin{aligned}\mathbb{E}[u_t^\varepsilon(x)^2] &= \mathbb{E}[(G_t * u_0^\varepsilon(x))^2] \\ &\qquad + \frac{\mathrm{Lip}(\sigma)^2}{\log \varepsilon^{-1}} \int_0^t \|\!|u_s^\varepsilon|\!\|^2 \iint G_{t-s}(x-y_1)G_{t-s}(x-y_2)G_{\varepsilon^2}(y_1-y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}s \\ &= \mathbb{E}[(G_t * u_0^\varepsilon(x))^2] + \frac{\mathrm{Lip}(\sigma)^2}{2\pi\log\varepsilon^{-1}} \int_0^t \frac{\|\!|u_s^\varepsilon|\!\|^2}{2(t-s)+\varepsilon^2}\,\mathrm{d}s. \end{aligned} \tag{13}$$

Since the right side does not depend on x, we in fact get

$$\|\!|u_t^\varepsilon|\!\|^2 \le \|\!|u_0|\!\|^2 + \frac{\mathrm{Lip}(\sigma)^2}{2\pi\log\varepsilon^{-1}} \int_0^t \frac{\|\!|u_s^\varepsilon|\!\|^2}{2(t-s)+\varepsilon^2}\,\mathrm{d}s.$$

Iterating this inequality, we get

$$\begin{aligned}\|\!|u_t^\varepsilon|\!\|^2 &\le \|\!|u_0|\!\|^2 \sum_{k=0}^\infty \left(\frac{\mathrm{Lip}(\sigma)^2}{2\pi\log\varepsilon^{-1}}\right)^k \iint_{0\le s_1\le\cdots\le s_k\le s_{k+1}=t} \prod_{i=1}^k \frac{1}{2(s_{i+1}-s_i)+\varepsilon^2}\,\mathrm{d}s \\ &\le \|\!|u_0|\!\|^2 \sum_{k=0}^\infty \left(\frac{\mathrm{Lip}(\sigma)^2}{2\pi\log\varepsilon^{-1}} \int_0^t \frac{1}{2s+\varepsilon^2}\,\mathrm{d}s\right)^k \\ &= \|\!|u_0|\!\|^2 \left(\frac{\mathrm{Lip}(\sigma)^2}{4\pi}\log_{\varepsilon^{-1}}(1+2t\varepsilon^{-2})\right)^k \\ &= \frac{\|\!|u_0|\!\|^2}{1-\frac{\mathrm{Lip}(\sigma)^2}{4\pi}\log_{\varepsilon^{-1}}(1+2t\varepsilon^{-2})} \\ &\to \frac{\|\!|u_0|\!\|^2}{1-\frac{\mathrm{Lip}(\sigma)^2}{2\pi}}\end{aligned}$$

as long as $\mathrm{Lip}(\sigma) < \sqrt{2\pi}$. □

Remark 2.3. The above proof only requires that $0 \le \sigma(u) \le \beta|u|$ for some $\beta < \sqrt{2\pi}$, rather than the Lipschitz condition.

2.2 The martingale

Now we see that our one-point fluctuations are tight. We'll now begin the process of trying to characterize the limiting law more precisely. We begin by fixing some $T \in (0,\infty)$ and defining

$$U_t^{T;\varepsilon}(x) = G_{T-t} * u_t^\varepsilon(x). \tag{14}$$

We record the following fact.

Lemma 2.4. *For fixed $T \in (0,\infty)$ and $X \in \mathbb{R}^2$, the process $(U_t^{T;\varepsilon}(X))_{t\in[0,T]}$ is a martingale.*

Proof. Combining (11) and (14), we have

$$U_t^{T;\varepsilon}(x) = G_T * u_0^\varepsilon(x) + \frac{1}{(\log \varepsilon^{-1})^{1/2}} \int_0^t G_{T-s} * \sigma(u_s^\varepsilon)\, \mathrm{d}W_s^\varepsilon.$$

The first term on the right side does not depend on t, and the integrand in the stochastic integral on the right side does not depend on t. Thus, it is at least a local martingale in t; it is not difficult to use the moment bound proved in Proposition 2.2 to see that its quadratic variation is bounded in L^2 and hence it's a martingale. □

Of course, whenever we have a martingale, we like to look at the quadratic variation. Actually, this was mentioned implicitly in the previous proof, but we note precisely that

$$\mathrm{d}[U^{T;\varepsilon}(x)]_t = \frac{1}{\log \varepsilon^{-1}} \int G_{T-t}(x-y_1) G_{T-t}(x-y_2) \sigma(u_t^\varepsilon(y_1)) \sigma(u_t^\varepsilon(y_2)) G_{\varepsilon^{-2}}(y_1 - y_2)\, \mathrm{d}y_1\, \mathrm{d}y_2. \tag{15}$$

Remark 2.5. Let's return briefly to the Edwards–Wilkinson setting. In that case, we have $\sigma(u) = \beta^2$, so this becomes

$$\mathrm{d}[U^{T;\varepsilon}(x)]_t = \frac{\beta^2}{2\pi(2(T-t)+\varepsilon^2)\log \varepsilon^{-1}}. \tag{16}$$

Thus, in this case, the martingale $(U_t^{T;\varepsilon}(x))_t$ is a time-changed Brownian motion, and we recover the Gaussian statistics as before. The variance at time T is in fact given by

$$[U^{T;\varepsilon}(x)]_T = \frac{\beta^2}{2\pi \log \varepsilon^{-1}} \int_0^T \frac{\mathrm{d}t}{2(T-t)+\varepsilon^2} = \frac{\beta^2}{4\pi} \log_{\varepsilon^{-1}}(1 + 2\varepsilon^{-2} T) \to \frac{\beta^2}{2\pi}$$

as above.

In the nonlinear setting, the quadratic variation (15) depends on the field u_t^ε. Ideally, we would write the right side of (15) as a function of $U_t^{T;\varepsilon}$: then, $U_t^{T;\varepsilon}$ would solve an SDE. However, the right side of (15) depends on not just the spatial average of u_t^ε but the average of the nonlinearity applied to u_t^ε. Thus it is not clear how we could derive a closed equation from (15). However, we will see later that the self-similar structure of the equation indeed allows us to derive such an equation, approximately.

2.3 Temporal discretization

It turns out to be helpful to discretize the time parameter t. We'll follow the discretization scheme given in [7, §6.1]. (For a continuous-time approach, see [6].) However, this discretization should of course be done according to the logarithmic time scale, such that on each discrete time step, the contribution of (16) is roughly the same. It is straightforward to see that this time scale is

$$t = T - \varepsilon^q, \qquad q \in (0, 2). \tag{17}$$

(The reason for the cutoff at $q = 2$ is the "$+\varepsilon^2$" appearing in (16). Then we can choose a parameter δ_ε such that

$$(\log \varepsilon^{-1})^{-1} \ll \delta_\varepsilon^2 \ll 1,$$

which means that we have both

$$\delta_\varepsilon \to 0 \qquad \text{and} \qquad \varepsilon^{\delta_\varepsilon} \to 0 \qquad \text{as } \varepsilon \to 0.$$

2.4 Turning off the noise

A key tool in our analysis will be an approximation scheme called *turning off the noise*. To carry this out, we first define another parameter γ_ε such that

$$(\log \varepsilon^{-1})^{-1} \ll \gamma_\varepsilon \ll \delta_\varepsilon^2 \ll 1.$$

Then we define

$$s_m = \varepsilon^{m\delta_\varepsilon}, \qquad s_m' = \varepsilon^{m\delta_\varepsilon + \gamma_\varepsilon}, \qquad t_m = T - s_m, \qquad t_{m'} = T - s_m'.$$

We will consider an approximation of the Markov chain $(U_{t_m}^{\varepsilon;T})_{m=1}^{\delta_\varepsilon^{-1}}$ and study its increments. To do this, we will consider the problem with the noise "turned off" on each interval $[t_m, t_m']$.

Let's first describe the intuition for this choice. As observed above in (17), the noise contributes on the scale $T - \varepsilon^q$. We have discretized q into $O(\delta_\varepsilon^{-1})$ chunks of length δ_ε. Since $\gamma_\varepsilon \ll \delta_\varepsilon$, we are turning off the noise only for a small chunk of time in terms of the q scale. On the other hand, when we consider the solution at time t_m' after the noise has just been turned off for time $t_m' - t_m = s_m - s_m' = \varepsilon^{m\delta_\varepsilon}(1 - \varepsilon^{\gamma_\varepsilon}) \approx \varepsilon^{m\delta_\varepsilon}$, the solution is actually smooth at spatial scale $\varepsilon^{m\delta_\varepsilon/2}$. (Recall that the scaling is parabolic so the spatial scale is the square root of the temporal scale.) Since the time remaining is only $T - t_m' = s_m' = \varepsilon^{m\delta_\varepsilon+\gamma_\varepsilon} \ll \varepsilon^{m\delta_\varepsilon}$, the solution at the final time T only effectively sees the solution at time t_m' via a constant. This gives us hope that we can reduce the problem down to a one-dimensional one.

Let's now an estimate about turning off the noise. From now on, let's drop the subscript ε on u_t^ε to save chalk.

Proposition 2.6 (See [7, Proposition 4.1] for a more general statement)**.** *Let $0 \le \tau_1 \le \tau_2 \le T$. There is a constant C such that the following holds. Suppose that u_t satisfies (7) and $\tilde{u}_t$ satisfies (7) for $t \in [0,\infty) \setminus [\tau_1, \tau_2]$ and just the ordinary heat equation*

$$\partial_t \tilde{u}_t(x) = \frac{1}{2}\Delta \tilde{u}_t(x) \qquad \text{for } t \in [\tau_1, \tau_2].$$

Then we have, for $t \ge \tau_2$,

$$\mathbb{E}\,(u_t(x) - \tilde{u}_t(x))^2 \le \frac{C}{\log \varepsilon^{-1}} \left(1 + \log \frac{t - \tau_1 + \varepsilon^2/2}{t - \tau_2 + \varepsilon^2/2}\right). \tag{18}$$

Proof. We note that $u_t = \tilde{u}_t$ whenever $t \le \tau_1$. Therefore, if we subtract two copies of the mild solution formula, we get

$$\begin{aligned} u_t(x) - \tilde{u}_t(x) &= \frac{1}{(\log \varepsilon^{-1})^{1/2}} \int_{\tau_1}^{\tau_2} \int G_{t-s}(x-y)\sigma(u_s(y))\,\mathrm{d}W_s^\varepsilon(y) \\ &\quad + \frac{1}{(\log \varepsilon^{-1})^{1/2}} \int_{\tau_2}^{t} \int G_{t-s}(x-y)\left(\sigma(u_s(y)) - \sigma(\tilde{u}_s(y))\right)\mathrm{d}W_s^\varepsilon(y). \end{aligned}$$

Then we can use the Itô isometry to write

$$
\begin{aligned}
&\mathbb{E}[u_t(x) - \tilde{u}_t(x)]^2 \\
&\quad = \frac{1}{\log \varepsilon^{-1}} \int_{\tau_1}^{\tau_2} \int G_{t-s}(x-y_1) G_{t-s}(x-y_2) \mathbb{E}[\sigma(u_s(y_1))\sigma(u_s(y_2))] G_{\varepsilon^2}(y_1 - y_2)\, \mathrm{d}y_1\, \mathrm{d}y_2\, \mathrm{d}s \\
&\qquad + \frac{1}{\log \varepsilon^{-1}} \int_{\tau_2}^{t} \int G_{t-s}(x-y_1) G_{t-s}(x-y_2) \mathbb{E}\left[\prod_{i=1}^{2} (\sigma(u_s(y_i)) - \sigma(\tilde{u}_s(y_i)))\right] G_{\varepsilon^2}(y_1 - y_2)\, \mathrm{d}y_1\, \mathrm{d}y_2\, \mathrm{d}s \\
&\quad \le \frac{\mathrm{Lip}(\sigma)^2}{\log \varepsilon^{-1}} \int_{\tau_1}^{\tau_2} |\!|\!| u_s |\!|\!| G_{2(t-s)+\varepsilon^2}(0)\, \mathrm{d}s + \frac{\mathrm{Lip}(\sigma)^2}{\log \varepsilon^{-1}} \int_{\tau_2}^{t} |\!|\!| u_s - \tilde{u}_s |\!|\!| G_{2(t-s)+\varepsilon^2}(0)\, \mathrm{d}s \\
&\quad = \frac{\mathrm{Lip}(\sigma)^2}{4\pi \log \varepsilon^{-1}} \int_{\tau_1}^{\tau_2} \frac{|\!|\!| u_s |\!|\!|}{t - s + \varepsilon^2/2}\, \mathrm{d}s + \frac{\mathrm{Lip}(\sigma)^2}{4\pi \log \varepsilon^{-1}} \int_{\tau_2}^{t} \frac{|\!|\!| u_s - \tilde{u}_s |\!|\!|}{t - s + \varepsilon^2/2}\, \mathrm{d}s.
\end{aligned}
$$

By Proposition 2.2, we can bound the first term by

$$\frac{C}{\log \varepsilon^{-1}} \int_{\tau_1}^{\tau_2} \frac{\mathrm{d}s}{t - s + \varepsilon^2/2} = \frac{C}{\log \varepsilon^{-1}} \log \frac{t - \tau_1 + \varepsilon^2/2}{t - \tau_2 + \varepsilon^2/2}.$$

Altogether, this means that if we define

$$f(t) \coloneqq |\!|\!| u_t - \tilde{u}_t |\!|\!|,$$

then for $t \in [\tau_2, T]$, we have,

$$f(t) \le \frac{C}{\log \varepsilon^{-1}} \log \frac{t - \tau_1 + \varepsilon^2/2}{t - \tau_2 + \varepsilon^2/2} + \frac{\mathrm{Lip}(\sigma)^2}{4\pi \log \varepsilon^{-1}} \int_{\tau_2}^{t} \frac{f(s)}{t - s + \varepsilon^2/2}\, \mathrm{d}s. \tag{19}$$

Now by a version of Grönwall's inequality **(see [7, Lemma 4.3])**, this implies (18). We note that again the condition $\mathrm{Lip}(\sigma) < \sqrt{2\pi}$ is key so that an infinite series is summable. □

Remark 2.7. We note that this bound works well in the case when $\tau_1 = t_m$, $\tau_2 = t'_m$, and $t \ge t_{m+1}$. For in that case, the right side of (18) is

$$\frac{C}{\log \varepsilon^{-1}} \left(1 + \log \frac{\varepsilon^{m\delta_\varepsilon} - O(\varepsilon^{(m+1)\delta_\varepsilon}) + \varepsilon^2/2}{\varepsilon^{m\delta_\varepsilon + \gamma_\varepsilon} - O(\varepsilon^{(m+1)\delta_\varepsilon}) + \varepsilon^2/2}\right) \approx \frac{C}{\log \varepsilon^{-1}} \log \varepsilon^{-\gamma_\varepsilon} = C\gamma_\varepsilon \to 0 \qquad \text{as } \varepsilon \to 0.$$

In fact, since $\gamma_\varepsilon^{1/2} \ll \delta_\varepsilon$, we can iterate the bound to turn off the noise on each of the intervals $[t_m, t'_m]$.

Remark 2.8. We will actually need a slightly stronger version of this lemma, where we allow for the noise to have already been turned off on some intervals (on both u_t and $\tilde{u}_t$). See [7, Proposition 4.1]. The proof is exactly the same; we omit the precise statement for notational simplicity.

2.5 Replacing the smoothed field with a constant

We now want to show that, if we turn off the noise on $[\tau_1, \tau_2]$, then we can replace the solution at time τ_2 by a constant equal to its value at some $X \in \mathbb{R}^2$ and again not change the solution at a later too much as long as we only consider the solution close to the point X. Applying this iteratively will allow us to replace the martingale by a one-dimensional Markov chain.

Proposition 2.9. *Fix $X \in \mathbb{R}^2$ and $T < \infty$. There is a constant $C = C(\sigma, u_0, T) < \infty$ such that the following holds. Suppose that $(\tilde{u}_t(x))$ is as in Proposition 2.6 and $\overline{u}_t(x)$ solves (7) for $t \in [\tau_2, T]$ with initial condition*

$$\overline{u}_{\tau_2}(x) = \tilde{u}_{\tau_2}(X),$$

so it has spatially constant initial condition at time τ_2. Then we have, for all $t \in [\tau_2, T]$, that

$$\mathbb{E}[(\overline{u}_t - \tilde{u}_t)(x)^2] \le C \cdot \frac{t - \tau_2 + |x - X|^2}{\tau_2 - \tau_1}. \tag{20}$$

Proof. We have

$$\tilde{u}_{\tau_2}(X) = \int G_{\tau_2 - \tau_1}(X - y)\tilde{u}_{\tau_1}(y)\,\mathrm{d}y,$$

since $(\tilde{u}_t)$ solves the deterministic heat equation on $[\tau_1, \tau_2]$ by construction. Therefore, for any $t > \tau_2$, we have

$$\begin{aligned}(\overline{u}_t - \tilde{u}_t)(x) &= \int \left[G_{\tau_2-\tau_1}(X - y) - G_{t-\tau_1}(x - y)\right]\tilde{u}_{\tau_1}(y)\,\mathrm{d}y \\ &\quad + \frac{1}{\sqrt{\log \varepsilon^{-1}}}\int_{\tau_2}^{t}\int G_{t-s}(x-y)[\sigma(\overline{u}_s(y)) - \sigma(\tilde{u}_s(y))]\,\mathrm{d}W_s^{\varepsilon}(y).\end{aligned}$$

We can take the second moment to obtain

$$\begin{aligned}&\mathbb{E}[(\overline{u}_t - \tilde{u}_t)(x)^2] \\ &\le \iint \mathbb{E}\left[\prod_{i=1}^{2}\left(\left[G_{\tau_2-\tau_1}(X - y_i) - G_{t-\tau_1}(x-y_i)\right]\tilde{u}_{\tau_1}(y)\right)\right]\mathrm{d}y_1\,\mathrm{d}y_2 \\ &\quad + \frac{\beta^2}{\log \varepsilon^{-1}}\int_{\tau_2}^{t}\iint G_{\varepsilon^2}(y_1 - y_2)\mathbb{E}\left[\prod_{i=1}^{2}\left(G_{t-s}(x-y_i)|\overline{u}_s(y_i) - \tilde{u}_s(y_i)|\right)\right]\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}s \\ &=: I_1 + I_2.\end{aligned} \tag{21}$$

For I_1, we use the Cauchy–Schwarz inequality on the probability space to write

$$\begin{aligned}I_1 &\le \left(\int \prod_{i=1}^{2}\left(\left[G_{\tau_2-\tau_1}(X-y_i) - G_{t-\tau_1}(x-y_i)\right]\left(\mathbb{E}\tilde{u}_{\tau_1}(y)^2\right)^{1/2}\right)\mathrm{d}y\right) \\ &\le C\|G_{\tau_2-\tau_1}(X - \cdot) - G_{t-\tau_1}(x - \cdot)\|_{L^1(\mathbb{R}^2)}^2 \le C\frac{t - \tau_2 + |X - x|^2}{\tau_2 - \tau_1},\end{aligned}$$

where we used the moment bound (Proposition 2.2), and then Pinsker's inequality; see [7, (5.8)] for details.

For I_2, we write using Young's inequality $|ab| \le \frac{1}{2}(a^2 + b^2)$ that

$$\begin{aligned}I_2 &\le \frac{\beta^2}{2\log \varepsilon^{-1}}\sum_{j=1}^{2}\int_{\tau_2}^{t}\iint G_{\varepsilon^2}(y_1 - y_2)\mathbb{E}|\overline{u}_s(y_j) - \tilde{u}_s(y_j)|^2\prod_{i=1}^{2}G_{t-s}(x-y_i)\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}s \\ &\le \frac{\beta^2}{\log \varepsilon^{-1}}\int_{\tau_2}^{t}\int G_{t-s+\varepsilon^2}(x-y)G_{t-s}(x-y)\mathbb{E}|\overline{u}_s(y) - \tilde{u}_s(y)|^2\,\mathrm{d}y\,\mathrm{d}s \\ &= \frac{\beta^2}{4\pi\log \varepsilon^{-1}}\int_{\tau_2}^{t}\frac{1}{t - s + \varepsilon^2/2}\int G_{\frac{(t-s)(t-s+\varepsilon^2)}{2(t-s)+\varepsilon^2}}(x-y)\mathbb{E}|\overline{u}_s(y) - \tilde{u}_s(y)|^2\,\mathrm{d}y\,\mathrm{d}s.\end{aligned}$$

Now we use the last two inequalities in (21) to get, if we define

$$f_t(x) = \mathbb{E}[(\overline{u}_t - \tilde{u}_t)(x)^2],$$

that

$$f_t(x) \le C\frac{t - \tau_2 + |X - x|^2}{\tau_2 - \tau_1} + \frac{\beta^2}{4\pi \log \varepsilon^{-1}} \int_{\tau_2}^{t} \frac{1}{t - s + \varepsilon^2/2} \int G_{\frac{(t-s)(t-s+\varepsilon^2)}{2(t-s)+\varepsilon^2}}(x - y) f_s(y)\,\mathrm{d}y\,\mathrm{d}s.$$

From this we use a Grönwall-type argument to derive (20). □

Remark 2.10. Again, this bound works well when $\tau_1 = t_m$, $\tau_2 = t'_m$, $t \ge t_{m+1}$, and $|x-X| \ll T - t_m$. In this setting, the right side of (20) becomes

$$C \cdot \frac{\varepsilon^{m\delta_\varepsilon+\gamma_\varepsilon} + |x - X|^2}{\varepsilon^{m\delta_\varepsilon} - \varepsilon^{m\delta_\varepsilon+\gamma_\varepsilon}} = C \cdot \frac{\varepsilon^{\gamma_\varepsilon} + \varepsilon^{-m\delta_\varepsilon}|x - X|^2}{1 - \varepsilon^{\gamma_\varepsilon}},$$

and this is small for small ε.

Remark 2.11. Similarly to the situation described in Remark 2.8, we will need a slightly stronger variant of this lemma that allows the noise to have also been turned off at certain points in the past.

2.6 Defining the Markov chain

Fix $T > 0$ and $X \in \mathbb{R}^2$. Choose some ζ_ε such that

$$(\log \varepsilon^{-1})^{-1} \ll \zeta_\varepsilon \ll 1$$

and then define

$$M_1 := \lceil \delta_\varepsilon^{-1} \log_\varepsilon T \rceil - 1;$$
$$M_2 := \lceil \delta_\varepsilon^{-1}(2 - \zeta_\varepsilon) \rceil,$$

which will be the starting and ending points of the Markov chain. If T is order 1, we have

$$M_1 \in \{-1, 0\} \qquad \text{for sufficiently small } \varepsilon,$$

and we note that

$$M_2 \approx 2\delta_\varepsilon^{-1} \qquad \text{for small } \varepsilon.$$

We now define a Markov chain $(Y_m)_{m=M_1}^{M_2}$. We give an informal description of it as follows. First, we define a process $(w_t(x))_{t\in[0,T],x\in\mathbb{R}^2}$ which satisfies (7), except that on intervals of the form $[t_m, t'_m]$ it instead satisfies the ordinary heat equation, and at each time t'_m the solution is replaced by the spatially-constant value $w_{t'_m}(X)$.

Lemma 2.12. *We have*

$$\lim_{\varepsilon\to 0} \mathbb{E}|Y_{M_2} - u_T(X)|^2 = 0.$$

The proof of this lemma is essentially an inductive application of Propositions 2.6 and 2.9, noting that the right sides are small by Remarks 2.7 and 2.10. The details take some time to work out, so we won't discuss them here and refer the reader to [7, Proposition 7.1].

Lemma 2.13. *The process $(Y_m)_{m=M_1}^{M_2}$ is a Markov chain and also a (discrete) martingale.*

This follows essentially from the definition and the fact that the evolution preserves the mean.

We have now discretized the problem and reduced it to studying a one-dimensional martingale. What remains is to determine the statistics of the increments of this martingale.

We define

$$J_\varepsilon(q,a) = \frac{1}{2\sqrt{\pi}} \left(\mathbb{E}[\sigma^2(u_{\varepsilon^{2-q}}(0)) \mid u_0 \equiv a]\right)^2.$$

Lemma 2.14. *The function $J_\varepsilon \colon [0,2]\times\mathbb{R}\to\mathbb{R}$ is equicontinuous on compact sets. Therefore, for any sequence $\varepsilon_k \downarrow 0$, there is a subsequence $\varepsilon_{k_n}\downarrow 0$ and a continuous function $J\colon [0,2]\times\mathbb{R}\to\mathbb{R}$ such that $J_\varepsilon \to J$ uniformly on compact sets.*

We note that

$$Y_{m+1} = Y_m + \frac{1}{\sqrt{\log\varepsilon^{-1}}}\int_{t'_m}^{t_m}\int G_{t'_{m+1}-t'_m}(X-y)\sigma(w_s(y))\,\mathrm{d}W_s^\varepsilon(y), \tag{22}$$

where $(w_t(x))$ satisfies (7) with initial condition

$$w_{t'_m}(x) = Y_m.$$

Then we can compute from (22) that

$$\begin{aligned}
&\mathbb{E}[Y_{m+1}^2 \mid Y_m = a] \\
&\quad= \frac{1}{\log\varepsilon^{-1}}\int_{t'_m}^{t_{m+1}}\int G_{t'_{m+1}-s}(X-y_1)G_{t'_{m+1}-s}(X-y_2)G_{\varepsilon^2}(y_1-y_2) \\
&\qquad\qquad\qquad\qquad \times \mathbb{E}\left[\sigma(w_s(y_1))\sigma(w_s(y_2))\right]\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}s \\
&\quad\approx \frac{1}{\log\varepsilon^{-1}}\int_{t'_m}^{t_{m+1}} J(2-\log_\varepsilon(t_{m+1}-s),a)\int G_{t'_{m+1}-s}(X-y)^2\,\mathrm{d}y\,\mathrm{d}s \\
&\quad\approx \frac{J(2-m\delta_\varepsilon,a)}{\log\varepsilon^{-1}}\int_{t'_m}^{t_{m+1}}\int G_{t'_{m+1}-s}(X-y)^2\,\mathrm{d}y\,\mathrm{d}s \\
&\quad\approx \delta_\varepsilon J_\varepsilon^2(2-m\delta_\varepsilon,a).
\end{aligned} \tag{23}$$

(For details on the approximations, see [7, Proposition 7.3].) We used in particular the equicontinuity of J_ε here.

Now, it is a consequence of Lemma 2.14 and (23) as well as [12, §11.2] then the process $(Y_{\lceil\delta_\varepsilon^{-1}q})_{q\in[0,2]}$ is approaching the solution to the SDE

$$\begin{aligned}
\mathrm{d}X(q) &= J(2-q,X(q))\mathrm{d}B(q); \\
X(0) &= a
\end{aligned}$$

along the subsequence $\varepsilon_k\downarrow 0$. (Recall that, at this point, J depends on the subsequence!)

At this point, we would really like to characterize the limiting object J, and in particular to show that it is unique and thus the convergence $J_\varepsilon \to J$ holds in reality, not simply along subsequences. But at this stage, we have not shown anything about this convergence. It is time to (finally) use the self-similarity of the problem!

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