Combinatorial Spanning Tree Models for Knot Homologies

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Knots in Washington XXXIII

Joint work with John Baldwin (Princeton University)

Spanning tree models for knot polynomials

Given a diagram D for a knot or link $K \subset S^3$, form the Tait graph or black graph $B(D)$:

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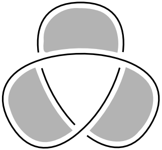

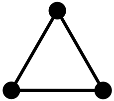

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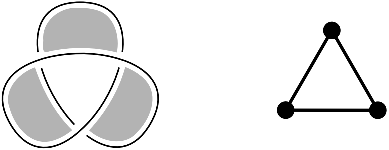
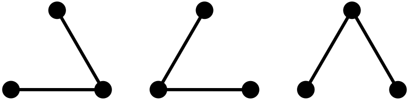


A *spanning tree* is a connected, simply connected subgraph of $B(D)$ containing all the vertices.



Spanning tree models for knot polynomials

The Alexander polynomial and Jones polynomials of K can be computed as sums of monomials corresponding to spanning trees: e.g.,

$$\Delta_K(t) = \sum_{s \in \text{Trees}(B(D))} (-1)^{a(s)} t^{b(s)}$$

where $a(s)$ and $b(s)$ are integers determined by s.

Knot Floer homology

Knot Floer homology (Ozsváth–Szabó, Rasmussen): for a link $K \subset S^3$, bigraded, finitely generated abelian group.

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- Detects the genus of the knot (Ozsváth–Szabó):

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- Detects fiberedness: K is fibered if and only if $\widehat{\mathrm{HFK}}_*(K, g(K)) \cong \mathbb{Z}$.

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- (Ozsváth–Szabó) There is a spectral sequence whose E_2 page is $\widetilde{\mathrm{Kh}}(\overline{K})$ and whose E_∞ page is $\widehat{\mathrm{HF}}(\Sigma(K))$, the Heegaard Floer homology of the branched double cover of K. Hence $\operatorname{rank} \widetilde{\mathrm{Kh}}(\overline{K}) \geq \operatorname{rank} \widehat{\mathrm{HF}}(\Sigma(K))$.

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- (Kronheimer–Mrowka) Similar spectral sequence from $\widetilde{\mathrm{Kh}}(K)$ to the instanton knot Floer homology of K, which detects the unknot. Hence $\widetilde{\mathrm{Kh}}(K) \cong \mathbb{Z}$ iff K is the unknot.

The δ grading

Often, it's helpful to collapse the two gradings into one, called the δ grading.

$$\widehat{\mathrm{HFK}}^{\delta}(K) = \bigoplus_{a-m=\delta} \widehat{\mathrm{HFK}}_m(K, a) \qquad \widetilde{\mathrm{Kh}}_{\delta}(K) = \bigoplus_{i-2j=\delta} \widetilde{\mathrm{Kh}}^{i,j}(K)$$

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Theorem (Manolescu–Ozsváth)

If K is a (quasi-)alternating link, then $\widehat{\mathrm{HFK}}(K;\mathbb{F})$ and $\widetilde{\mathrm{Kh}}(K;\mathbb{F})$ are both supported in a single δ grading, namely $\delta = -\sigma(K)/2$, where $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

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Conjecture

For any ℓ-component link K,

$$2^{\ell-1} \operatorname{rank} \widetilde{\mathrm{Kh}}_{\delta}(K;\mathbb{F}) \geq \operatorname{rank} \widehat{\mathrm{HFK}}^{\delta}(K;\mathbb{F}).$$

Spanning tree complexes

Can we find explicit spanning tree complexes for $\widehat{\mathrm{HFK}}(K)$ and $\widetilde{\mathrm{Kh}}(K)$? Specifically, want to find a complex C such that:

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Theorem (Baldwin–L., Roberts, Jaeger, Manion)

Yes.

Earlier results

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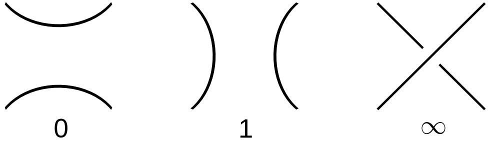
- Wehrli and Champarnerkar-Kofman showed that the standard Khovanov complex reduces to a complex generated by spanning trees, but they weren't able to describe the differential explicitly.

Cube of resolutions

Label the crossings $c_1, \ldots, c_n$. For $I = (i_1, \ldots, i_n) \in \{0,1\}^n$, let D_I be the diagram gotten by taking the i_j-resolution of c_j:



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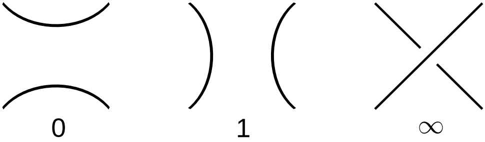

Let $|I| = i_1 + \cdots + i_n$, and let $\ell_I =$ be the number of components of D_I.

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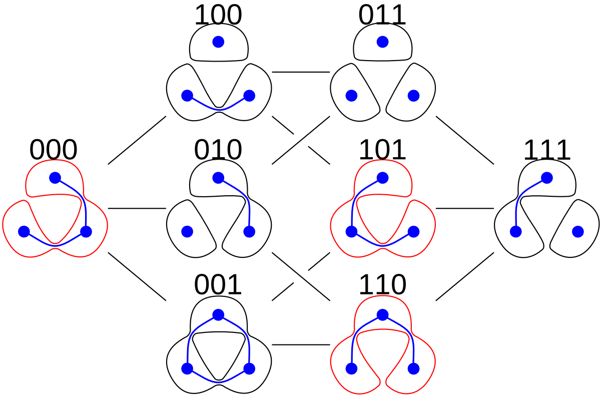

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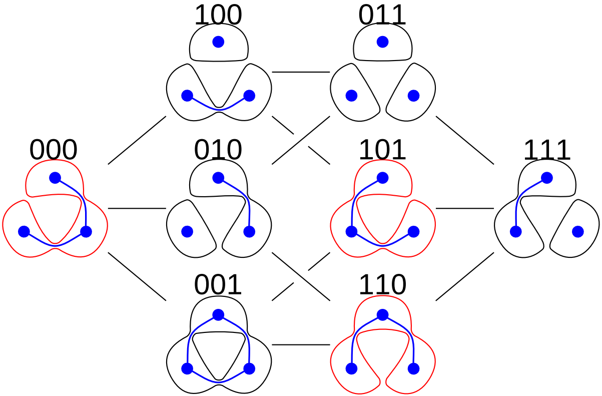

Let $R(D) = \{I \in \{0,1\}^n \mid \ell_I = 1\}$. For $I, I' \in R(D)$, we say I' is a double successor of I if I' is gotten by changing two 0s to 1s.

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- Label the edges of D $e_1, \ldots, e_{2n}$. For each $I \in R(D)$, we define Y_I to be a vector space over $\mathbb{F}(T)$ with generators $y_1, \ldots, y_{2n}$, satisfying a single linear relation whose coefficients are powers of T depending on the order in which $e_1, \ldots, e_{2n}$ occur in D_I.

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- Let
$$C(D) = \bigoplus_{I \in R(D)} \Lambda^*(Y_I).$$
Declare the grading of $\Lambda^*(Y_I)$ to be $\frac{1}{2}(|I| - n_-(D))$.

Spanning tree model for $\widehat{\text{HFK}}$

For each double successor pair, we define a linear map

$$f_{I,I'} \colon \Lambda^*(Y_I) \to \Lambda^*(Y_{I'}),$$

which is (almost always) a vector space isomorphism. Let

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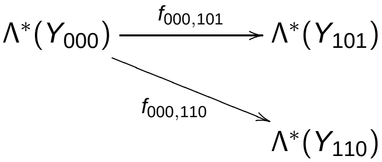

$\mathrm{gr} = -1$ $\qquad$ $\mathrm{gr} = 0$

Spanning tree model for $\widehat{\mathrm{HFK}}$

Theorem (Baldwin–L. 2011)

For any diagram D of an ℓ-component link K, $(C(D), \partial_D)$ is a chain complex, and

$$H_*(C(D), \partial_D) \cong \widehat{\mathrm{HFK}}(K; \mathbb{F}) \otimes \mathbb{F}(T)^{2n-\ell}$$

where $\widehat{\mathrm{HFK}}(K)$ is equipped with its δ grading.

Spanning tree model for Kh

- Roberts defined a complex consisting of a copy of $\mathbb{F}(X_1, \dots, X_{2n})$ for each $I \in R(D)$, and a nonzero differential for each double successor pair I, I', which is multiplication by some element of the field determined by the two two-component resolutions in between I and I'. The grading is the same as in our complex.

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- Jaeger proved that when K is a knot, the homology of this complex is $\widetilde{\mathrm{Kh}}(K;\mathbb{F}) \otimes \mathbb{F}(X_1, \dots, X_{2n})$, with its δ grading.

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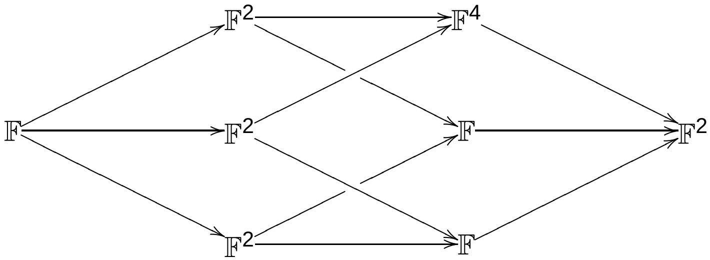
- Manion showed how to do this with coefficients in $\mathbb{Z}$ rather than $\mathbb{F}$. The resulting homology theory is odd Khovanov homology.

Khovanov homology

Khovanov associates a vector space V_I of dimension $2^{\ell_I - 1}$ to each resolution, and a map $d_{I,I'}\colon V_I \to V_I'$ whenever I' is an immediate successor of I. Let ∂_{Kh} be the differential of this complex.



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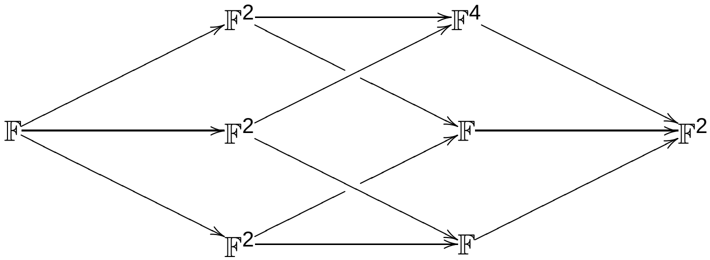


$\widetilde{\mathrm{Kh}}(K)$ is defined to be $H_*(\partial_{\mathrm{Kh}})$.

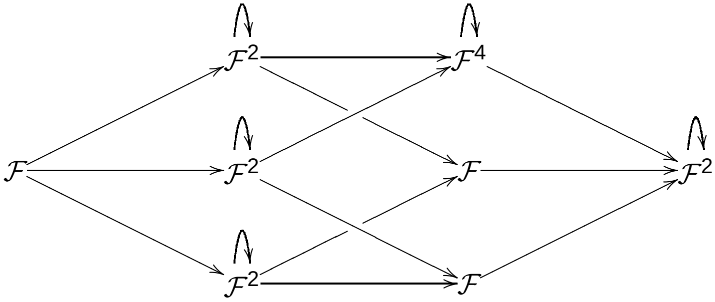
Twisted Khovanov homology

Roberts: Let $\mathcal{F} = \mathbb{F}(X_1, \dots, X_{2n})$, and let $\mathcal{V}_I = V_I \otimes \mathcal{F}$. Define an internal differential ∂_I on $\mathcal{V}_I$ such that

$$H_*(\mathcal{V}_I, \partial_I) = \begin{cases} \mathcal{V}_I & \ell_I = 1 \\ 0 & \ell_I > 1. \end{cases}$$

Let $\partial_V = \sum_I \partial_I$. By choosing ∂_I carefully, we can arrange that $\partial_V \partial_{\text{Kh}} = \partial_{\text{Kh}} \partial_V$, so that $(\partial_V + \partial_{\text{Kh}})^2 = 0$.

Twisted Khovanov homology



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Roberts showed that the resulting homology is a link invariant. Jaeger showed that if K is a knot, this homology is isomorphic to $\widetilde{\text{Kh}}(K) \otimes \mathcal{F}$.

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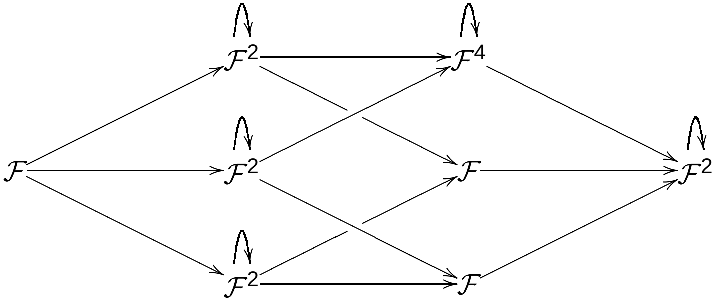

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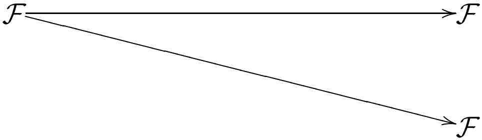

Cube of resolutions for $\widehat{\mathrm{HFK}}$

Let V be a $\mathbb{F}$-vector space of rank 2. Manolescu showed that there is an unoriented skein sequence for $\widehat{\mathrm{HFK}}$:

$$\begin{array}{ccc} \widehat{\mathrm{HFK}}(K) \otimes V^{\otimes m-\ell} & \longrightarrow & \widehat{\mathrm{HFK}}(K_0) \otimes V^{\otimes m-\ell_0} \\ & \nwarrow \quad \swarrow & \\ & \widehat{\mathrm{HFK}}(K_1) \otimes V^{\otimes m-\ell_1} & \end{array}$$

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Essentially, we need these extra powers of V because $\widehat{\mathrm{HFK}}$ of a link is "too big." For example, $\widehat{\mathrm{HFK}}$ of the Hopf link has rank 4, while both resolutions at a crossing are unknots, for which $\widehat{\mathrm{HFK}}$ has rank 1. This is the big difference between $\widehat{\mathrm{HFK}}$ and other invariants ($\widetilde{\mathrm{Kh}}(K)$, $\widehat{\mathrm{HF}}(\Sigma(K))$, instanton knot Floer homology, etc.)

Cube of resolutions of $\widehat{\mathrm{HFK}}$

Iterating this (à la Ozsváth–Szabó), we get a cube of resolutions for $\widehat{\mathrm{HFK}}$: a differential on

$$\bigoplus_{I \in \{0,1\}^n} \widehat{\mathrm{HFK}}(K_I) \otimes V^{m-\ell_I}$$

consisting of a sum of maps

$$f_I \colon \widehat{\mathrm{HFK}}(K_I) \otimes V^{\otimes m-\ell_I} \to \widehat{\mathrm{HFK}}(K_{I'}) \otimes V^{\otimes m-\ell_{I'}}$$

for every pair I, I', whose homology is $\widehat{\mathrm{HFK}}(K) \otimes V^{\otimes m-\ell}$.

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- Can also do something similar for the spectral sequence from $\widetilde{\text{Kh}}(K)$ to $\widehat{\text{HF}}(\Sigma(-K))$. The only problem is that we don't have the grading argument that would imply the spectral sequence collapses after E_2. But E_3 is an invariant (Kriz–Kriz).