Non-orientable Surfaces in 3- and 4-Manifolds

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Introduction

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Theorem (L.–Ruberman–Strle)

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Question

For each homology class in $x \in H_2(M;\mathbb{Z})$, what is the minimal complexity of an embedded surface representing x?

- $n = 4$: can always find connected surfaces, so complexity just means genus.
- $n = 3$: have to be a bit careful about how to handle disconnected surfaces. Thurston semi-norm on $H_2(M;\mathbb{Q})$.

Minimal genus problems in three dimensions

- If $M = \Sigma_g \times S^1$, or more generally any Σ_g bundle over S^1, the homology class $[\Sigma_g \times \{\text{pt}\}]$ cannot be represented by a surface of lower genus. (Elementary algebraic topology.)

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- If Σ is a leaf of a taut foliation on M, then Σ minimizes complexity in its homology class (Thurston, 1970s).
- If $\Sigma \subset M$ minimizes complexity in its homology class, then there exists a taut foliation on M of which Σ is a leaf (Gabai, 1980s).

Minimal genus problems in four dimensions

- In $\mathbb{CP}^2$, the solution set of a generic homogenous polynomial of degree d is a surface of genus $(d-1)(d-2)/2$, representing d times a generator of $H_2(\mathbb{CP}^2; \mathbb{Z})$.

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Theorem (Thom conjecture: Kronheimer–Mrowka, 1994)

If $\Sigma \subset \mathbb{CP}^2$ is a surface of genus g representing d times a generator, then

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Theorem (Symplectic Thom conjecture: Ozsváth–Szabó, 2000)

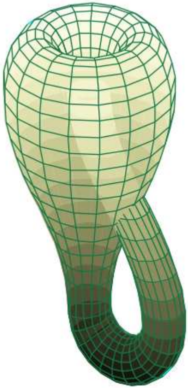
If X is a symplectic 4-manifold, and $\Sigma \subset X$ is a symplectic surface, then Σ minimizes genus in its homology class.

Non-orientable surfaces

- Let

$$F_h = \underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{h \text{ copies}},$$

the non-orientable surface of genus h.



(Image credit: Wikipedia)

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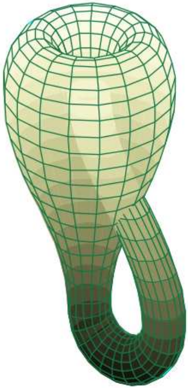

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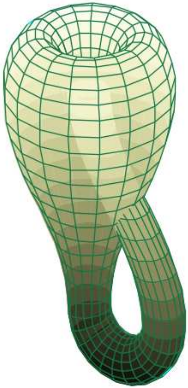
- For any M of dimension 3 or 4, any class in $H_2(M; \mathbb{Z}_2)$ can be represented by a non-orientable surface.
- An embedding $F_h \subset M^3$ must represent a nontrivial class in $H_2(M; \mathbb{Z}_2)$. In particular, no F_h embeds in $\mathbb{R}^3$, but any F_h can be immersed in $\mathbb{R}^3$.



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- A standard $\mathbb{RP}^2 \subset \mathbb{R}^4$ has Euler number ± 2. The connected sum of h of these has Euler number in

$$\{-2h, -2h+4, \dots, 2h-4, 2h\}.$$

Non-orientable surfaces

Theorem (Massey, 1969; conjectured by Whitney, 1940)

For any embedding of F_h in $\mathbb{R}^4$ (or S^4, or any homology 4-sphere) with normal Euler number e, we have

$$|e| \leq 2h \quad \textit{and} \quad e \equiv 2h \pmod 4.$$

Non-orientable surfaces in lens spaces

- For p, q relatively prime, the lens space $L(p, q)$ is the quotient of

$$S^3 = \left\{ (z, w) \in \mathbb{C}^2 \;\middle|\; |z|^2 + |w|^2 = 1 \right\}$$

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- Alternate description: glue together two copies of $S^1 \times D^2$ via a gluing map that takes $\{\text{pt}\} \times \partial D^2$ in one copy to a curve homologous to

$$p[S^1 \times \{\text{pt}\}] + q[\{\text{pt}\} \times \partial D^2]$$

in the other copy.

Non-orientable surfaces in lens spaces

Theorem (Bredon–Wood)

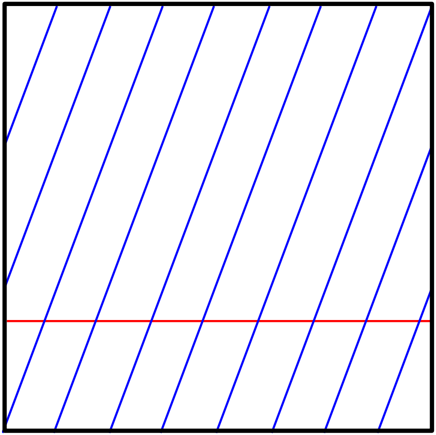
If F_h embeds in the lens space $L(2k,q)$, then $h = N(2k,q) + 2i$, where $i \geq 0$ and

- $N(2,1) = 1$;
- $N(2k,q) = N(2(k-q),q') + 1$, *where* $q' \in \{1,\dots,k-q\}$, $q' \equiv \pm q \pmod{2(k-q)}$.

Moreover, all such values of h are realizable.

Non-orientable surfaces in lens spaces

It's quite easy to see the minimal genus surfaces. For instance, $N(8,3) = N(2,1) + 1 = 2$.



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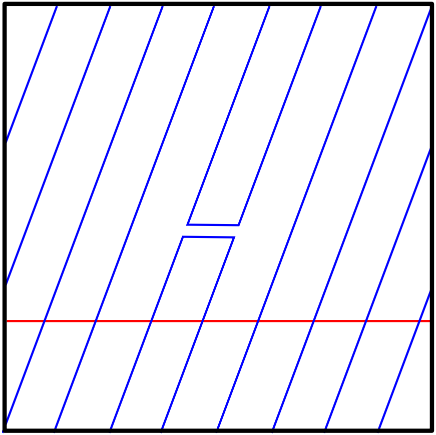

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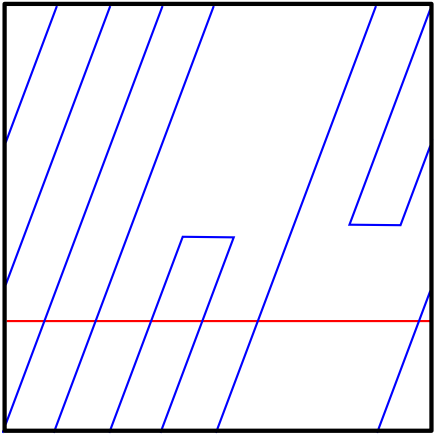

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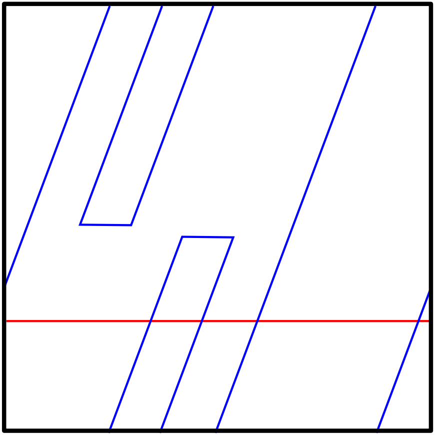

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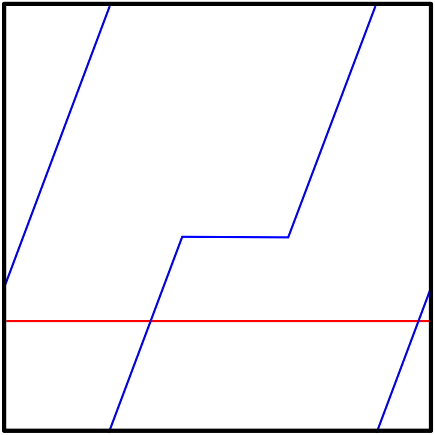

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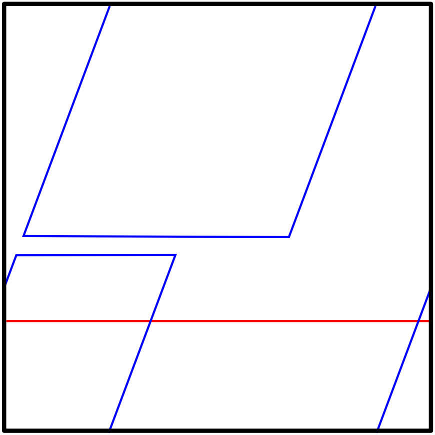

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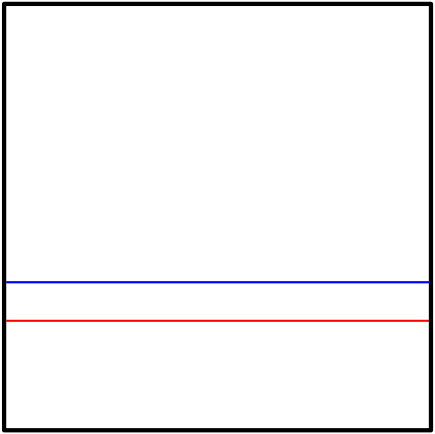

Non-orientable surfaces in 3-manifolds

Question

Can we find a general framework for genus bounds for embeddings of non-orientable surfaces in other 3-manifolds?

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- To any closed, oriented, connected 3-manifold M, associate a $\mathbb{Z}[U]$–module $\mathrm{HF}^+(M)$.
- $\mathrm{HF}^+(M)$ splits as a direct sum

$$\mathrm{HF}^+(M) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(M)} \mathrm{HF}^+(M, \mathfrak{s}).$$

  corresponding to the set of spin^c structures on M, which is an affine set for $H^2(M; \mathbb{Z})$. All but finitely many summands are 0.

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- A cobordism W from M_0 to M_1, equipped with a spin^c structure $\mathfrak{t}$, induces a map

$$F^+_{W,\mathfrak{t}} \colon \mathrm{HF}^+(M_0, \mathfrak{t}|_{M_0}) \to \mathrm{HF}^+(M_1, \mathfrak{t}|_{M_1}).$$

Heegaard Floer homology

- When $H_2(M;\mathbb{Z}) \neq 0$, the set of spin^c structures on M for which $\mathrm{HF}^+(M,\mathfrak{s}) \neq 0$ completely determines the Thurston norm of M, i.e., the minimal complexity of embedded surfaces representing any homology class in $H_2(M;\mathbb{Z})$ (Ozsváth–Szabó 2004).

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- The groups $\mathrm{HF}^+(M,\mathfrak{s})$ also determine which homology classes in $H_2(M;\mathbb{Z})$, if any, can be represented by the fiber of a fibration over S^1.

Heegaard Floer homology

- If M is a rational homology sphere (i.e. $H_1(M;\mathbb{Z})$ finite, $H_2(M;\mathbb{Z}) = 0$), there are finitely many spin^c structures, and

$$\mathrm{HF}^+(M,\mathfrak{s}) \cong \mathbb{Z}[U,U^{-1}]/\mathbb{Z}[U] \oplus \text{f. g. abelian group}$$

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- M is called an L-space if $\mathrm{HF}^+(M)$ is as small as possible:

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  for each $\mathfrak{s}$.
  - Examples: S^3, lens spaces, any M with finite π_1, branched double covers of (quasi)-alternating links in S^3.

Non-orientable surfaces in 3-manifolds

If M is a rational homology sphere, and $x \in H_2(M;\mathbb{Z}_2)$:

- Let φ_x be the image of x under

$$H_2(M;\mathbb{Z}_2) \xrightarrow{\beta} H_1(M;\mathbb{Z}) \xrightarrow{\text{PD}} H^2(M;\mathbb{Z}),$$

  where β is the Bockstein homomorphism. This is an element of order 2 (unless $x = 0$).

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  - E.g., if $H_1(M;\mathbb{Z}) \cong H^2(M;\mathbb{Z}) \cong \mathbb{Z}/2k$, and x is the nonzero element of $H_2(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$, then $\varphi_x = k$.

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- Let

$$\Delta(M,x) = \max_{\mathfrak{s} \in \text{Spin}^c(M)} (d(M,\mathfrak{s}+\varphi_x) - d(M,\mathfrak{s})) \in \frac{1}{2}\mathbb{Z}.$$

Theorem (via Ni–Wu, 2012)

If M is a rational homology sphere with $H_1(M;\mathbb{Z}) \cong H^2(M;\mathbb{Z}) \cong \mathbb{Z}/2k$, and F_h embeds in M, then

$$h \geq 2\Delta(M,[F_h]).$$

Furthermore, if M is an L-space and there is a Floer simple knot representing the class $k \in H_1(M;\mathbb{Z})$, then there exists an embedding $F_h \hookrightarrow M$ yielding equality above.

Non-orientable surfaces in lens spaces

Corollary

For the lens space $L(2k, q)$,

$$N(2k, q) = 2\Delta(L(2k, q), x).$$

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(With Ira Gessel) Can show using Dedekind sums that the RHS satisfies the same recursion as $N(2k, q)$, giving a new proof of Bredon–Wood.

Non-orientable surfaces in 4-manifolds

Question

Can we do better in 4 dimensions? For instance, can we find an embedding of F_h in $M \times I$ that's not allowed in M?

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- More generally, can consider not just $M \times I$, but any homology cobordism between rational homology spheres M_0 and M_1: a compact, oriented 4-manifold W with $\partial W = -M_0 \sqcup M_1$, such that the inclusions $M_i \hookrightarrow W$ induce isomorphisms on homology.

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- If M_0 and M_1 are homology cobordant, then they have the same d-invariants (Ozsváth–Szabó).

Non-orientable surfaces in 4-manifolds

Theorem (L.–Ruberman–Strle)

Let $W: M_0 \to M_1$ be a homology cobordism between rational homology spheres, and suppose that F_h embeds in W with normal Euler number e. Let $\Delta = \Delta(M_0, [F]) = \Delta(M_1, [F])$. Then

$$h \geq 2\Delta, \quad |e| \leq 2h - 4\Delta, \quad \text{and} \quad e \equiv 2h - 4\Delta \pmod 4.$$

Corollary

If F_h embeds in $L(2k,q) \times I$ (or, more generally, in any homology cobordism from $L(2k,q)$ to itself) with normal Euler number e, representing a nontrivial $\mathbb{Z}_2$ homology class, then

$$h \geq N(2k,q) \quad \text{and} \quad |e| \leq 2(h - N(2k,q)).$$

In other words, F_h has the same genus and normal Euler number as a stabilization of an embedding in $L(2k,q)$.

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- More generally, if M is an L-space and contains a Floer-simple knot in each order-2 homology class, then the minimal genus problem in $M \times I$ (or any homology cobordism from M to itself) is the same as the minimal genus problem in M.

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- What if we only require the surfaces to be topologically locally flat, not smoothly embedded? Or only require the homology cobordisms to be topological manifolds?
  - If $M = S^3_{+2}(D_+(T_{2,3}))$, then there is a topological homology cobordism from M to itself that contains an embedded $\mathbb{RP}^2$, but this 4-manifold can't be smoothed.
  - Is there a locally flat $\mathbb{RP}^2$ in $L(4,1) \times I$?

Non-orientable surfaces in closed, definite 4-manifolds

Theorem (L.–Ruberman–Strle)

Suppose X is a closed, positive definite 4-manifold with $H_1(X;\mathbb{Z}) = 0$, and F_h embeds in X with normal Euler number e. Denote by ℓ the minimal self-intersection of an integral lift of $[F_h]$. Then

$$e \equiv \ell - 2h \pmod 4 \quad \textit{and} \quad e \geq \ell - 2h.$$

Additionally, if $\ell = b_2(X)$, then

$$e \leq 9b + 10h - 16.$$