Non-surjective satellite operators and piecewise-linear concordance

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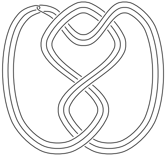

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- Equivalently, which knots in $\mathbb{R}^3$ (or S^3) bound properly embedded disks in $\mathbb{R}^4_+$ (or D^4)?

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$$\mathcal{C} = \{\text{knots}\}/\text{smooth conc.} \qquad \mathcal{C}^{\text{top}} = \{\text{knots}\}/\text{top. conc.}$$

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Conjecture (Zeeman, 1963)

In an arbitrary compact, contractible 4-manifold X other than the 4-ball, not every knot $K \subset \partial X$ bounds a PL disk.

Piecewise-linear concordance

Theorem (Matsumoto–Venema, 1979)

There exists a non-compact, contractible 4-manifold with boundary $S^1 \times \mathbb{R}^2$ such that $S^1 \times \{\mathrm{pt}\}$ does not bound an embedded PL disk.

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Theorem (Akbulut, 1990)

There exist a compact, contractible 4-manifold X and a knot $\gamma \subset \partial X$ that does not bound an embedded PL disk in X.

Akbulut's example

- Akbulut's manifold X is the original Mazur manifold:

$$X = S^1 \times D^3 \cup_Q \text{2-handle},$$
$$Q \subset S^1 \times D^2 \subset \partial(S^1 \times D^3),$$
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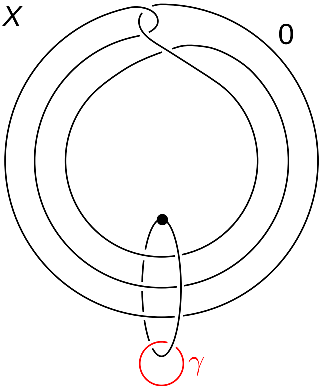

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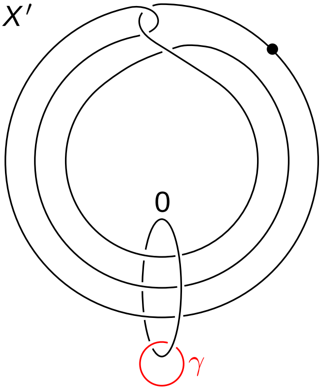

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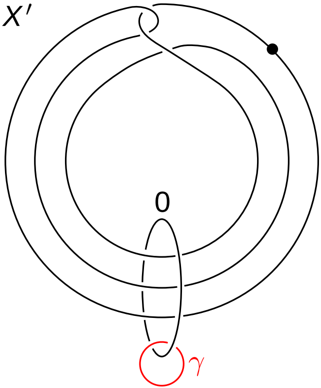

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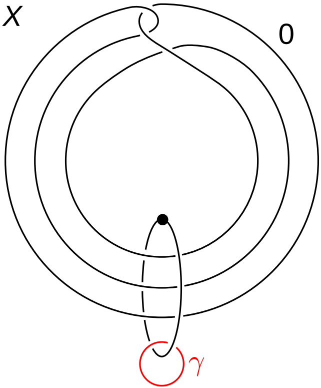
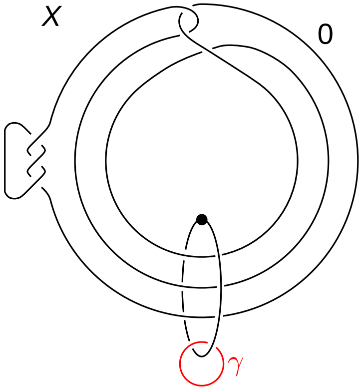


Main theorem

Theorem (L., 2014)

There exist a contractible 4-manifold X and a knot $\gamma \subset \partial X$ such that γ does not bound an embedded PL disk in any contractible manifold X' with $\partial X' = \partial X$.



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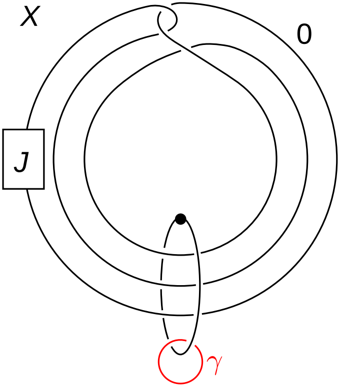
- In place of the trefoil, can use any knot J with $\epsilon(J) = 1$, where ϵ is Hom's concordance invariant.



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Freedman: If $\Delta_K(t) \equiv 1$, then K is topologically slice; e.g., Whitehead doubles. But many such knots are not smoothly slice.

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  - $\mathcal{C}/\ker(\epsilon)$ contains a $\mathbb{Z}^\infty$ summand of topologically slice knots.

Expanded notions of smooth concordance

- Every knot $K \subset S^3$ bounds a smooth disk in some 4-manifold X with $\partial X = S^3$; for instance, can take $X = (k\,\mathbb{C}P^2 \# l\,\overline{\mathbb{C}P^2}) \setminus B^4$.

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- Let $\mathcal{C}_R$ and $\mathcal{C}_{\text{ex}}$ denote the corresponding concordance groups, so that

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- Classical obstructions, Heegaard Floer obstructions all vanish if K is $\mathbb{Z}$–homology slice.
- Rasmussen's invariant $s(K)$ (coming from Khovanov homology) was originally only proven to obstruct honest smooth concordance, but Kronheimer and Mrowka showed it actually descends to $\mathcal{C}_{\text{ex}}$.

Expanded notions of concordance

Definition

Knots K_1, K_2 in homology spheres Y_1, Y_2 are

- *R*–homology concordant if there is a smooth R-homology cobordism W from Y_1 to Y_2 (i.e. $H_*(Y_i; R) \xrightarrow{\cong} H_*(W; R)$) and a smooth annulus in W connecting K_1 and K_2;

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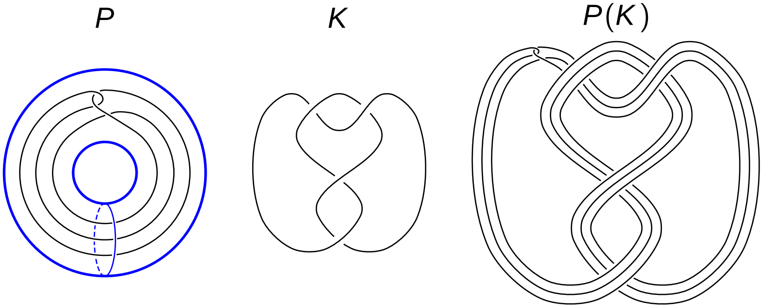
- A knot $K \subset Y$ bounds a PL disk in a contractible 4-manifold X iff it is exotically cobordant to a knot in S^3, since we can delete a ball containing all the singularities.

Satellite operators

Definition

Given a pattern knot $P \subset S^1 \times D^2$ and a companion knot $K \subset S^3$, the satellite knot $P(K) \subset S^3$ is the image of P under the Seifert framing $S^1 \times D^2 \hookrightarrow S^3$ of K.



Satellite operators

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- Satellite operators are generally not group homomorphisms.

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- $P \subset S^1 \times D^2$ has winding number n if it represents n times a generator of $H_1(S^1 \times D^2)$.
- P has strong winding number 1 if the meridian $[\{\text{pt}\} \times \partial D^2]$ normally generates $\pi_1(S^1 \times D^2 \smallsetminus P)$.

Non-surjective satellite operators

Theorem (L., 2014)

There exists a (strong) winding number 1 pattern $P \subset S^1 \times D^2$ such that $P(K)$ is not $\mathbb{Z}$–homology slice for any knot $K \subset S^3$ (including the unknot).

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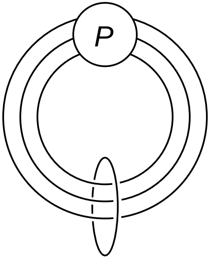
- It suffices to find a pattern Q such that $Q\colon \mathcal{C}^{\mathbb{Z}} \to \mathcal{C}^{\mathbb{Z}}$ is not surjective, and set $P = Q \# -J$ for $J \notin \operatorname{im}(Q)$.

Proof of the main theorem

- Let P be a winding number 1 pattern such that $P(K)$ is not $\mathbb{Z}$–homology slice for any K.



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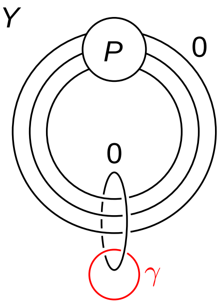
- Let P be a winding number 1 pattern such that $P(K)$ is not $\mathbb{Z}$–homology slice for any K.
- Let Y be the boundary of the Mazur-type manifold obtained from P, and let γ be the knot $S^1 \times \{pt\}$.



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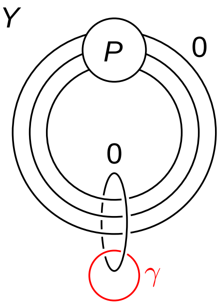
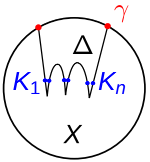
- Let Y be the boundary of the Mazur-type manifold obtained from P, and let γ be the knot $S^1 \times \{\text{pt}\}$.
- Suppose γ bounds a PL disk Δ in a contractible 4-manifold X with $\partial X = Y$. Can assume that Δ has singularities that are cones on knots $K_1, \dots, K_n \subset S^3$.





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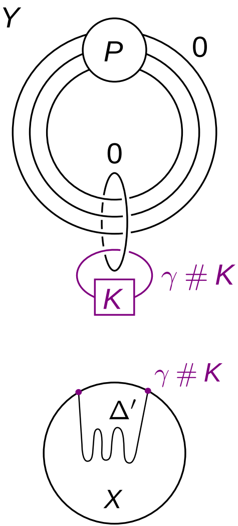

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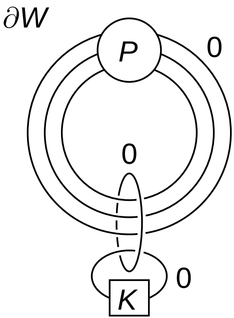
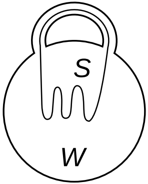
- Attach a 0-framed 2-handle along $\gamma \# K$ to obtain W, a homology $S^2 \times D^2$, whose H_2 is generated by an embedded sphere S with trivial normal bundle.





Proof of the main theorem

- Drill out arcs to see that $\gamma \# K$ bounds a smooth slice disk $\Delta' \subset X$, where $K = -(K_1 \# \cdots \# K_n)$.
- Attach a 0-framed 2-handle along $\gamma \# K$ to obtain W, a homology $S^2 \times D^2$, whose H_2 is generated by an embedded sphere S with trivial normal bundle.
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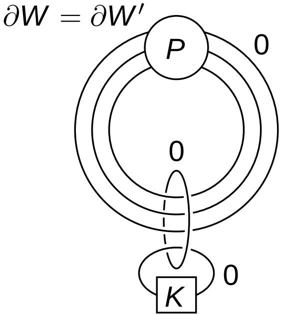

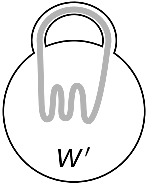

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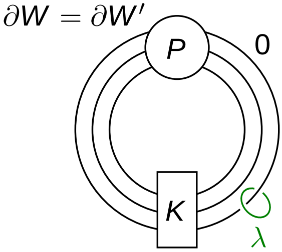
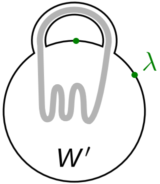
- Now $\partial W = \partial W' \cong S^3_0(P(K))$, and $H_1(W')$ is generated by λ.





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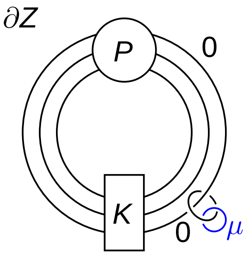

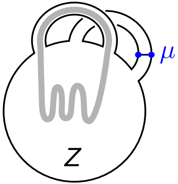

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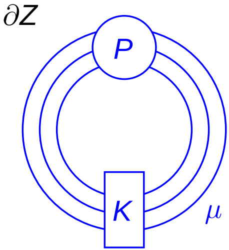
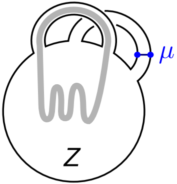
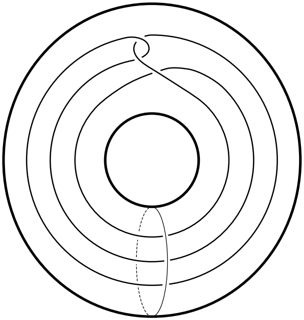
- Attach a 0-framed 2-handle along λ to obtain Z, a homology D^4. The belt circle μ of this 2-handle bounds a smoothly embedded disk (the cocore).
- The boundary of Z is S^3, and $\mu = P(K)$. Contradiction!





Non-surjective satellite operators

Let Q denote the Mazur pattern:



Non-surjective satellite operators

Proposition

For any knot $K \subset S^3$,

$$\tau(Q(K)) = \begin{cases} \tau(K) & \text{if } \tau(K) \leq 0 \text{ and } \epsilon(K) \in \{0,1\} \\ \tau(K)+1 & \text{if } \tau(K) > 0 \text{ or } \epsilon(K) = -1 \end{cases}$$

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- Proof uses bordered Floer homology, with computations assisted by Bohua Zhan's Python implementation of Lipshitz, Ozsváth, Thurston's arc slides algorithm.

Corollary

For any knot K and any $m > 1$, $\tau(Q^m(K)) \notin \{1, \ldots, m-1\}$.

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- Hence, the iterates of Q are decreasing self-similarities of $\mathcal{C}^{\text{ex}}$.