Khovanov homology and knot Floer homology for pointed links

John A. Baldwin, Adam Simon Levine, and Sucharit Sarkar

AMS Southeastern Sectional Meeting
November 13, 2016

Main conjecture

Conjecture (Rasmussen, Baldwin–L.)

With coefficients in any field $\mathbb{F}$, for any l-component link $L \subset S^3$ equipped with a basepoint $p \in L$, we have

$$2^{l-1} \operatorname{rank} \widetilde{\mathrm{Kh}}(L, p; \mathbb{F}) \geq \operatorname{rank} \widehat{\mathrm{HFK}}(L; \mathbb{F})$$

Main conjecture

Conjecture (Rasmussen, Baldwin–L.)

With coefficients in any field $\mathbb{F}$, for any l-component link $L \subset S^3$ equipped with a basepoint $p \in L$, we have

$$2^{l-1} \operatorname{rank} \widetilde{\mathrm{Kh}}(L, p; \mathbb{F}) \geq \operatorname{rank} \widehat{\mathrm{HFK}}(L; \mathbb{F})$$

- $\widetilde{\mathrm{Kh}}(L, p)$ denotes the reduced (even) Khovanov homology of L.
- $\widehat{\mathrm{HFK}}(L)$ denotes the knot Floer homology of L.

Main conjecture

Conjecture (Rasmussen, Baldwin–L.)

With coefficients in any field $\mathbb{F}$, for any l-component link $L \subset S^3$ equipped with a basepoint $p \in L$, we have

$$2^{l-1} \operatorname{rank} \widetilde{\mathrm{Kh}}(L, p; \mathbb{F}) \geq \operatorname{rank} \widehat{\mathrm{HFK}}(L; \mathbb{F})$$

- $\widetilde{\mathrm{Kh}}(L, p)$ denotes the reduced (even) Khovanov homology of L.
- $\widehat{\mathrm{HFK}}(L)$ denotes the knot Floer homology of L.
- Henceforth, we will work over $\mathbb{F} = \mathbb{Z}_2$.

Known results

For a link $L \subset S^3$, there are spectral sequences from $\widetilde{\mathrm{Kh}}(L)$ to many familiar invariants:

- Heegaard Floer homology of the branched double cover (Ozsváth–Szabó)

Known results

For a link $L \subset S^3$, there are spectral sequences from $\widetilde{\mathrm{Kh}}(L)$ to many familiar invariants:

- Heegaard Floer homology of the branched double cover (Ozsváth–Szabó)
- Instanton knot homology (Kronheimer–Mrowka)

Known results

For a link $L \subset S^3$, there are spectral sequences from $\widetilde{\mathrm{Kh}}(L)$ to many familiar invariants:

- Heegaard Floer homology of the branched double cover (Ozsváth–Szabó)
- Instanton knot homology (Kronheimer–Mrowka)

Corollary

Khovanov homology detects the unknot.

Known results

For a link $L \subset S^3$, there are spectral sequences from $\widetilde{\mathrm{Kh}}(L)$ to many familiar invariants:

- Heegaard Floer homology of the branched double cover (Ozsváth–Szabó)
- Instanton knot homology (Kronheimer–Mrowka)

Corollary

Khovanov homology detects the unknot.

- Monopole Floer homology of the branched double cover (Bloom)
- Instanton Floer homology of the branched double cover (Scaduto)
- Plane Floer homology (Daemi)
- Szabó homology

Skein exact sequences

Let A denote any of the invariants above. Basic properties:

- If L is an L-component unlink, then

$$A(L) \cong \widetilde{\mathrm{Kh}}(L) \cong \mathbb{F}^{2^{l-1}}.$$

Skein exact sequences

Let A denote any of the invariants above. Basic properties:

- If L is an L-component unlink, then

$$A(L) \cong \widetilde{\mathrm{Kh}}(L) \cong \mathbb{F}^{2^{l-1}}.$$

- The maps on A induced by elementary merges and splits agree with those on $\widetilde{\mathrm{Kh}}$.

Skein exact sequences

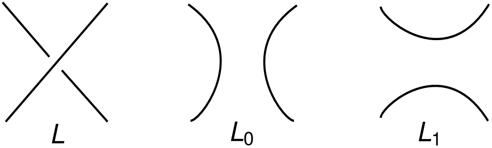
Let A denote any of the invariants above. Basic properties:

- If L is an L-component unlink, then

$$A(L) \cong \widetilde{\mathrm{Kh}}(L) \cong \mathbb{F}^{2^{l-1}}.$$

- The maps on A induced by elementary merges and splits agree with those on $\widetilde{\mathrm{Kh}}$.
- There is a skein sequence

$$\cdots \to A(L) \to A(L_0) \to A(L_1) \to A(L) \to \ldots,$$



Cube spectral sequences

For an n-crossing link diagram, L, generalize the construction of the skein sequence to obtain a filtered chain complex:

$$X_A(L) = \bigoplus_{v \in \{0,1\}^n} C_A(L_v) \qquad\qquad D = \sum_{v \leq v'} d_{v,v'}$$

with

$$H_*(X_A(L), D) = A(L).$$

Cube spectral sequences

For an n-crossing link diagram, L, generalize the construction of the skein sequence to obtain a filtered chain complex:

$$X_A(L) = \bigoplus_{v \in \{0,1\}^n} C_A(L_v) \qquad D = \sum_{v \leq v'} d_{v,v'}$$

with

$$H_*(X_A(L), D) = A(L).$$

Filtering by cube position, we obtain a spectral sequence:

$$E_1(X_A(L), D) = \bigoplus_{v \in \{0,1\}^n} A(L_v) = \widetilde{\mathrm{CKh}}(L)$$

$$E_2(X_A(L), D) = \widetilde{\mathrm{Kh}}(L)$$

Cube spectral sequences

For an n-crossing link diagram, L, generalize the construction of the skein sequence to obtain a filtered chain complex:

$$X_A(L) = \bigoplus_{v \in \{0,1\}^n} C_A(L_v) \qquad D = \sum_{v \leq v'} d_{v,v'}$$

with

$$H_*(X_A(L), D) = A(L).$$

Filtering by cube position, we obtain a spectral sequence:

$$E_1(X_A(L), D) = \bigoplus_{v \in \{0,1\}^n} A(L_v) = \widetilde{\mathrm{CKh}}(L)$$

$$E_2(X_A(L), D) = \widetilde{\mathrm{Kh}}(L)$$

Therefore,

$$\operatorname{rank} \widetilde{\mathrm{Kh}}(L) \geq \operatorname{rank} A(L).$$

Knot Floer homology

Knot Floer homology fails to satisfy this skein sequence. For instance, if $L =$ Hopf link, and $L_0 = L_1 =$ unknot,

$$\operatorname{rank} \widehat{\mathrm{HFK}}(L) = 4 \qquad \operatorname{rank} \widehat{\mathrm{HFK}}(L_0) = \operatorname{rank} \widehat{\mathrm{HFK}}(L_1) = 1$$

which can't fit into an exact triangle.

Knot Floer homology

Knot Floer homology fails to satisfy this skein sequence. For instance, if $L =$ Hopf link, and $L_0 = L_1 =$ unknot,

$$\operatorname{rank} \widehat{\mathrm{HFK}}(L) = 4 \qquad \operatorname{rank} \widehat{\mathrm{HFK}}(L_0) = \operatorname{rank} \widehat{\mathrm{HFK}}(L_1) = 1$$

which can't fit into an exact triangle.

Definition

A *pointed link* is $\mathcal{L} = (L, \mathbf{p})$, where $\mathbf{p}$ is a finite set of points on L. $\mathcal{L}$ is *nondegenerate* if every component of L contains at least one point of $\mathbf{p}$.

Knot Floer homology

- Knot Floer homology is really an invariant of *non-degenerate pointed links*.

$$\widehat{\mathrm{HFK}}(L) = \widehat{\mathrm{HFK}}(L, \{\text{one point on each component}\}).$$

Knot Floer homology

- Knot Floer homology is really an invariant of *non-degenerate pointed links*.

$$\widehat{\mathrm{HFK}}(L) = \widehat{\mathrm{HFK}}(L, \{\text{one point on each component}\}).$$

- If $p_0 \in \mathbf{p}$ is on the same component as some other point of $\mathbf{p}$, then

$$\widehat{\mathrm{HFK}}(L, \mathbf{p}) \cong \widehat{\mathrm{HFK}}(L, \mathbf{p} \setminus \{p_0\}) \otimes V, \qquad V = \mathbb{F}_{(0,0)} \oplus \mathbb{F}_{(-1,-1)}.$$

Knot Floer homology

- Knot Floer homology is really an invariant of *non-degenerate pointed links*.

$$\widehat{\mathrm{HFK}}(L) = \widehat{\mathrm{HFK}}(L, \{\text{one point on each component}\}).$$

- If $p_0 \in \mathbf{p}$ is on the same component as some other point of $\mathbf{p}$, then

$$\widehat{\mathrm{HFK}}(L, \mathbf{p}) \cong \widehat{\mathrm{HFK}}(L, \mathbf{p} \setminus \{p_0\}) \otimes V, \qquad V = \mathbb{F}_{(0,0)} \oplus \mathbb{F}_{(-1,-1)}.$$

- For a pointed link $\mathcal{L}$, let $\hat{\mathcal{L}}$ be the split union of $\mathcal{L}$ with an unknot with one marked point. Then

$$\widehat{\mathrm{HFK}}(\hat{\mathcal{L}}) \cong \widehat{\mathrm{HFK}}(\mathcal{L}) \otimes W, \qquad W = \mathbb{F}_{(1/2,0)} \oplus \mathbb{F}_{(-1/2,0)}.$$

We will think of this as "unreduced" $\widehat{\mathrm{HFK}}$.

Knot Floer homology

- Manolescu: if $\mathbf{p}$ is taken such that $(L, \mathbf{p})$, $(L_0, \mathbf{p})$, and $(L_1, \mathbf{p})$ are all nondegenerate, have a skein sequence:

$$\cdots \to \widehat{\mathrm{HFK}}(-L, \mathbf{p}) \to \widehat{\mathrm{HFK}}(-L_0, \mathbf{p}) \to \widehat{\mathrm{HFK}}(-L_1, \mathbf{p}) \to \cdots .$$

Knot Floer homology

- Manolescu: if $\mathbf{p}$ is taken such that $(L, \mathbf{p})$, $(L_0, \mathbf{p})$, and $(L_1, \mathbf{p})$ are all nondegenerate, have a skein sequence:

$$\cdots \to \widehat{\mathrm{HFK}}(-L, \mathbf{p}) \to \widehat{\mathrm{HFK}}(-L_0, \mathbf{p}) \to \widehat{\mathrm{HFK}}(-L_1, \mathbf{p}) \to \cdots .$$

- If L is an n-crossing, l-component link diagram, and $\mathbf{p}$ is a set of m points with at least one point on each edge, we can iterate to get a filtered complex

$$X(\mathcal{L}) = \bigoplus_{v \in \{0,1\}^n} \widehat{\mathrm{CFK}}(-\mathcal{L}_v) \qquad D = \sum_{v \leq v'} d_{v,v'}$$

with

$$H_*(X(\mathcal{L}), D) \cong \widehat{\mathrm{HFK}}(\mathcal{L}) \cong \widehat{\mathrm{HFK}}(L) \otimes V^{\otimes(m-l)}.$$

Knot Floer homology

- If we filter $X(\mathcal{L})$ by cube position:

$$E_1(X(\mathcal{L}), D) = \bigoplus_{v \in 0,1^n} \widehat{\mathrm{HFK}}(-\mathcal{L}_v)$$

$$= \bigoplus_{v \in \{0,1\}^n} W^{\otimes l_v - 1} \otimes V^{\otimes m - l_v}$$

$$= \bigoplus_{v \in \{0,1\}^n} \mathbb{F}^{2^{m-1}}$$

This looks very different from the Khovanov complex!

Knot Floer homology

- If we filter $X(\mathcal{L})$ by cube position:

$$E_1(X(\mathcal{L}), D) = \bigoplus_{v \in 0,1^n} \widehat{\mathrm{HFK}}(-\mathcal{L}_v)$$
$$= \bigoplus_{v \in \{0,1\}^n} W^{\otimes l_v - 1} \otimes V^{\otimes m - l_v}$$
$$= \bigoplus_{v \in \{0,1\}^n} \mathbb{F}^{2^{m-1}}$$

This looks very different from the Khovanov complex!

- The E_2 page isn't an invariant.

Knot Floer homology

- If we filter $X(\mathcal{L})$ by cube position:

$$E_1(X(\hat{\mathcal{L}}), D) = \bigoplus_{v \in 0,1^n} \widehat{\mathrm{HFK}}(-\hat{\mathcal{L}}_v)$$
$$= \bigoplus_{v \in \{0,1\}^n} W^{\otimes l_v} \quad \otimes V^{\otimes m - l_v}$$
$$= \bigoplus_{v \in \{0,1\}^n} \mathbb{F}^{2^m}$$

  This looks very different from the Khovanov complex!
- The E_2 page isn't an invariant.
- It will be more convenient to do everything with $\hat{\mathcal{L}}$.

Khovanov homology for pointed links

- Given a pointed link $(L, \mathbf{p})$, consider the unreduced Khovanov complex $(\mathrm{CKh}(L), d_{\mathrm{Kh}})$. For each $p \in \mathbf{p}$, have a chain map

  $$\xi_p\colon \ \mathrm{CKh}(L) \to \mathrm{CKh}(L),$$

  and these maps commute and square to 0.

Khovanov homology for pointed links

- Given a pointed link $(L, \mathbf{p})$, consider the unreduced Khovanov complex $(\mathrm{CKh}(L), d_{\mathrm{Kh}})$. For each $p \in \mathbf{p}$, have a chain map
$$\xi_p\colon\ \mathrm{CKh}(L) \to \mathrm{CKh}(L),$$
and these maps commute and square to 0.
- Let $\Lambda_{\mathbf{p}} = \Lambda^*(y_p \mid p \in \mathbf{p})$, and define
$$\mathrm{CKh}(L, \mathbf{p}) = \mathrm{CKh}(L) \otimes \Lambda_{\mathbf{p}} \qquad d = d_{\mathrm{Kh}} \otimes 1 + \sum_{p \in \mathbf{p}} \xi_p \otimes y_p.$$
Let
$$\mathrm{Kh}(L, \mathbf{p}) = H_*(\mathrm{CKh}(L, \mathbf{p}), d).$$

Khovanov homology for pointed links

- Given a pointed link $(L, \mathbf{p})$, consider the unreduced Khovanov complex $(\mathrm{CKh}(L), d_{\mathrm{Kh}})$. For each $p \in \mathbf{p}$, have a chain map
$$\xi_p \colon \mathrm{CKh}(L) \to \mathrm{CKh}(L),$$
and these maps commute and square to 0.
- Let $\Lambda_{\mathbf{p}} = \Lambda^*(y_p \mid p \in \mathbf{p})$, and define
$$\mathrm{CKh}(L, \mathbf{p}) = \mathrm{CKh}(L) \otimes \Lambda_{\mathbf{p}} \qquad d = d_{\mathrm{Kh}} \otimes 1 + \sum_{p \in \mathbf{p}} \xi_p \otimes y_p.$$
Let
$$\mathrm{Kh}(L, \mathbf{p}) = H_*(\mathrm{CKh}(L, \mathbf{p}), d).$$
- I'm being lazy about signs; all of this works over $\mathbb{Z}$.

Khovanov homology for pointed links

Theorem

Let $\mathcal{L} = (L, \mathbf{p})$ *be a pointed link in* S^3*, where* $|\mathbf{p}| = m > 0$*.*

- $\mathrm{Kh}(L, \mathbf{p})$ *is a pointed link invariant.*

Khovanov homology for pointed links

Theorem

Let $\mathcal{L} = (L, \mathbf{p})$ be a pointed link in S^3, where $|\mathbf{p}| = m > 0$.

- $\mathrm{Kh}(L, \mathbf{p})$ *is a pointed link invariant.*
- *If* $\mathbf{p}$ *contains a point p that is on the same component of L as some other point of* $\mathbf{p}$*, then*

$$\mathrm{Kh}(L, \mathbf{p}) \cong \mathrm{Kh}(L, \mathbf{p} \setminus \{p\}) \otimes V.$$

Khovanov homology for pointed links

Theorem

Let $\mathcal{L} = (L, \mathbf{p})$ *be a pointed link in* S^3*, where* $|\mathbf{p}| = m > 0$*.*

- $\mathrm{Kh}(L, \mathbf{p})$ *is a pointed link invariant.*
- *If* $\mathbf{p}$ *contains a point* p *that is on the same component of* L *as some other point of* $\mathbf{p}$*, then*

$$\mathrm{Kh}(L, \mathbf{p}) \cong \mathrm{Kh}(L, \mathbf{p} \setminus \{p\}) \otimes V.$$

- *With coefficients in any field* $\mathbb{F}$*, and for each point* $p \in \mathbf{p}$*,*

$$\operatorname{rank} \mathrm{Kh}(L, \mathbf{p}; \mathbb{F}) \leq 2^m \operatorname{rank} \widetilde{\mathrm{Kh}}(L, p; \mathbb{F}).$$

When L is a knot this relation is an equality.

Khovanov homology for pointed links

Theorem

Let $\mathcal{L} = (L, \mathbf{p})$ *be a pointed link in* S^3*, where* $|\mathbf{p}| = m > 0$*.*

- $\mathrm{Kh}(L, \mathbf{p})$ *is a pointed link invariant.*
- *If* $\mathbf{p}$ *contains a point* p *that is on the same component of* L *as some other point of* $\mathbf{p}$*, then*

$$\mathrm{Kh}(L, \mathbf{p}) \cong \mathrm{Kh}(L, \mathbf{p} \setminus \{p\}) \otimes V.$$

- *With coefficients in any field* $\mathbb{F}$*, and for each point* $p \in \mathbf{p}$*,*

$$\operatorname{rank} \mathrm{Kh}(L, \mathbf{p}; \mathbb{F}) \leq 2^m \operatorname{rank} \widetilde{\mathrm{Kh}}(L, p; \mathbb{F}).$$

  When L *is a knot this relation is an equality.*
- *If* L *is an unlink, then* $\mathrm{Kh}(L, \mathbf{p})$ *is canonically isomorphic to* $\widehat{\mathrm{HFK}}(\hat{\mathcal{L}})$*, with rank* 2^m*.*

Relating the theories

- There is an additional filtration on $(X(\hat{\mathcal{L}}), D)$ coming from the internal Alexander gradings on the summands $\widehat{\mathrm{CFK}}(\hat{\mathcal{L}}_v)$. Let D^0 denote the associated graded differential, so that

$$\operatorname{rank} H_*(X(\hat{\mathcal{L}}), D^0) \geq \operatorname{rank} H_*(X(\hat{\mathcal{L}}), D) = 2^{m-l+1} \operatorname{rank} \widehat{\mathrm{HFK}}(L).$$

Relating the theories

- There is an additional filtration on $(X(\hat{\mathcal{L}}), D)$ coming from the internal Alexander gradings on the summands $\widehat{\mathrm{CFK}}(\hat{\mathcal{L}}_v)$. Let D^0 denote the associated graded differential, so that

  $$\operatorname{rank} H_*(X(\hat{\mathcal{L}}), D^0) \geq \operatorname{rank} H_*(X(\hat{\mathcal{L}}), D) = 2^{m-l+1} \operatorname{rank} \widehat{\mathrm{HFK}}(L).$$

- We conjecture that

  $$H_*(X(\hat{\mathcal{L}}), D^0) \cong \mathrm{Kh}(L, \mathbf{p}; \mathbb{Z}_2),$$

  which has rank $\leq 2^m \operatorname{rank} \widetilde{\mathrm{Kh}}(L; \mathbb{Z}_2)$.

Relating the theories

- There is an additional filtration on $(X(\hat{\mathcal{L}}), D)$ coming from the internal Alexander gradings on the summands $\widehat{\mathrm{CFK}}(\hat{\mathcal{L}}_v)$. Let D^0 denote the associated graded differential, so that

$$\operatorname{rank} H_*(X(\hat{\mathcal{L}}), D^0) \geq \operatorname{rank} H_*(X(\hat{\mathcal{L}}), D) = 2^{m-l+1} \operatorname{rank} \widehat{\mathrm{HFK}}(L).$$

- We conjecture that

$$H_*(X(\hat{\mathcal{L}}), D^0) \cong \mathrm{Kh}(L, \mathbf{p}; \mathbb{Z}_2),$$

which has rank $\leq 2^m \operatorname{rank} \widetilde{\mathrm{Kh}}(L; \mathbb{Z}_2)$.

- This would imply that

$$2^{l-1} \operatorname{rank} \widetilde{\mathrm{Kh}}(L; \mathbb{Z}_2) \geq \operatorname{rank} \widehat{\mathrm{HFK}}(L).$$

Relating the theories

- Both $(X(\hat{\mathcal{L}}), D^0)$ and $(\mathrm{CKh}(\mathcal{L}), d)$ are filtered by cube position, giving rise to spectral sequences.

Relating the theories

- Both $(X(\hat{\mathcal{L}}), D^0)$ and $(\mathrm{CKh}(\mathcal{L}), d)$ are filtered by cube position, giving rise to spectral sequences.

Proposition

The E^1 pages of the cube filtrations of $(X(\hat{\mathcal{L}}), D^0)$ and $(\mathrm{CKh}(\mathcal{L}), d)$ are naturally isomorphic as chain complexes.

Relating the theories

- Both $(X(\hat{\mathcal{L}}), D^0)$ and $(\mathrm{CKh}(\mathcal{L}), d)$ are filtered by cube position, giving rise to spectral sequences.

Proposition

The E^1 pages of the cube filtrations of $(X(\hat{\mathcal{L}}), D^0)$ and $(\mathrm{CKh}(\mathcal{L}), d)$ are naturally isomorphic as chain complexes.

- Goal: Construct a filtered chain map on the total complexes realizing the isomorphism on the E^1 page. This would imply the conjecture.

Relating the theories

- Both $(X(\hat{\mathcal{L}}), D^0)$ and $(\mathrm{CKh}(\mathcal{L}), d)$ are filtered by cube position, giving rise to spectral sequences.

Proposition

The E^1 pages of the cube filtrations of $(X(\hat{\mathcal{L}}), D^0)$ and $(\mathrm{CKh}(\mathcal{L}), d)$ are naturally isomorphic as chain complexes.

- Goal: Construct a filtered chain map on the total complexes realizing the isomorphism on the E^1 page. This would imply the conjecture.
- Problem is that this requires understanding all of the holomorphic polygons that go into the definition of D^0.