Ribbon Concordance and Link Homology Theories

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Concordance

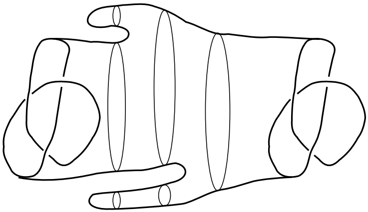
- Given knots $K_0, K_1 \subset S^3$, a concordance from K_0 to K_1 is a smoothly embedded annulus $A \subset S^3 \times [0,1]$ with

$$\partial A = -K_0 \times \{0\} \cup K_1 \times \{1\}.$$

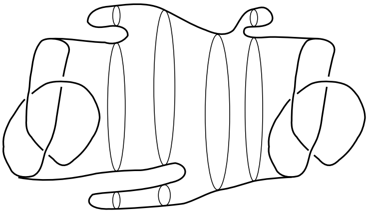
  K_0 and K_1 are called concordant ($K_0 \sim K_1$) if such a concordance exists.
- $\sim$ is an equivalence relation.
- K is slice if it is concordant to the unknot — or equivalently, if it bounds a smoothly embedded disk in D^4.
- For links L_0, L_1 with the same number of components, a concordance is a disjoint union of concordances between the components. L is (strongly) slice if it is concordant to the unlink.

Ribbon concordance

- A concordance $A \subset S^3 \times [0,1]$ from L_0 to L_1 is called a ribbon concordance if projection to $[0,1]$, restricted to A, is a Morse function with only index 0 and 1 critical points. We say L_0 is ribbon concordant to L_1 ($L_0 \preceq L_1$) if a ribbon concordance exists.

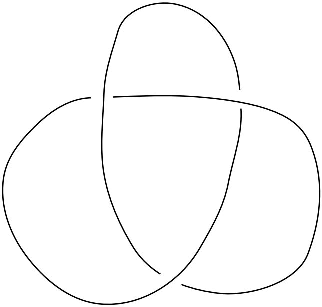


Ribbon

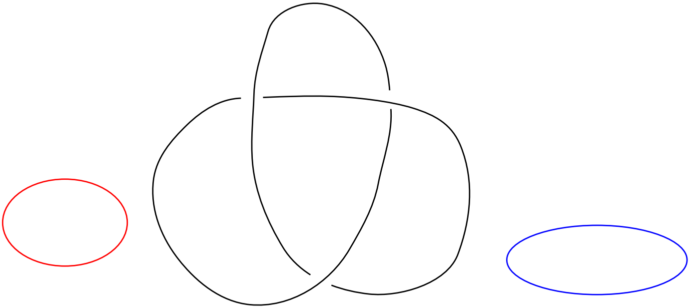


Not ribbon

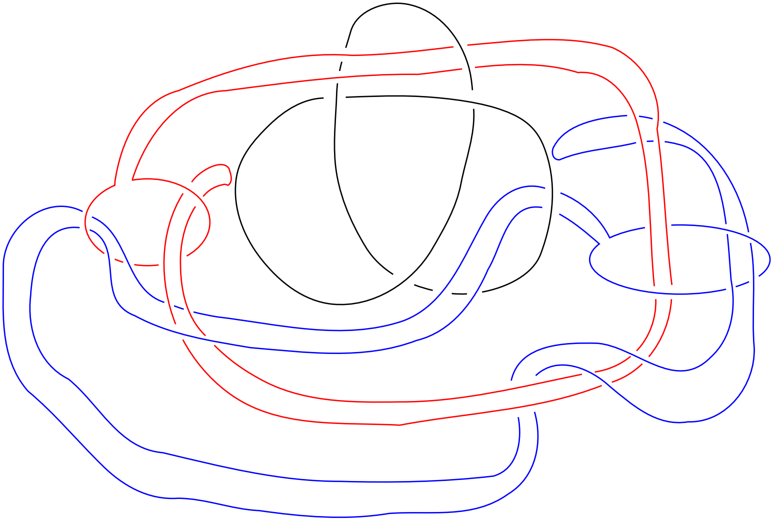
Ribbon concordance



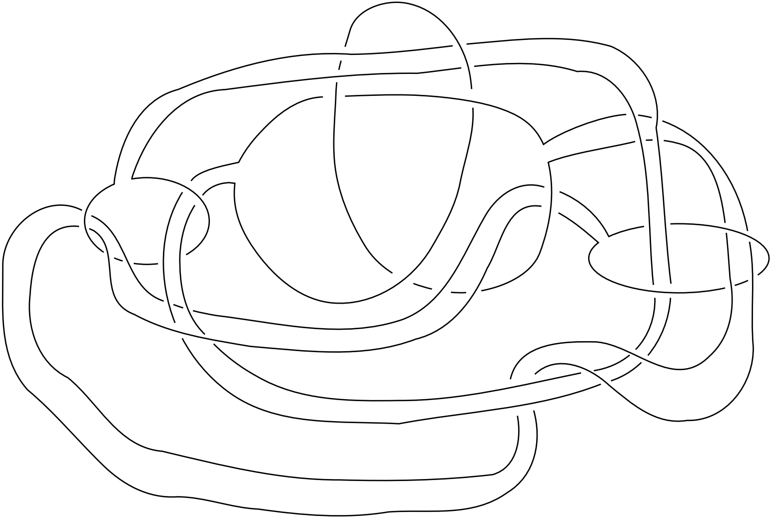
Ribbon concordance



Ribbon concordance



Ribbon concordance



Ribbon concordance

- K is a *ribbon knot* if the unknot is ribbon concordant to K; this is equivalent to bounding a slice disk in D^4 for which the radial function has only 0 and 1 critical points.

Conjecture (Slice-ribbon conjecture)

Every slice knot is ribbon.

- The above terminology is backwards from Gordon's original definition, where "from" and "to" are reversed. (But his $\preceq$ is the same.)

Ribbon concordance

- Ribbon concordance is reflexive and transitive, but definitely not symmetric!

Conjecture (Gordon 1981)

If K_0, K_1 are knots in S^3 such that $K_0 \preceq K_1$ and $K_1 \preceq K_0$, then K_0 and K_1 are isotopic ($K_0 = K_1$).
I.e., $\preceq$ is a partial order on the set of isotopy classes of knots.

- Philosophy: If $L_0 \preceq L_1$, then L_0 is "simpler" than L_1. And if $L_0 \preceq L_1$ and $L_1 \preceq L_0$, then lots of invariants cannot distinguish L_0 and L_1.

Ribbon concordance and π_1

Let C be a concordance from L_0 to L_1.

- If C is ribbon, with r births, then

$$\begin{aligned}(S^3 \times [0,1]) - \mathrm{nbd}(C) \\ &\cong (S^3 - \mathrm{nbd}(L_0)) \times [0,1] \cup (r \text{ 1-handles}) \cup (r \text{ 2-handles}) \\ &\cong (S^3 - \mathrm{nbd}(L_1)) \times [0,1] \cup (r \text{ 2-handles}) \cup (r \text{ 3-handles}).\end{aligned}$$

  (C is strongly homotopy ribbon.)

- This implies:

$$\pi_1(S^3 - L_0) \hookrightarrow \pi_1(S^3 \times [0,1] - C) \twoheadleftarrow \pi_1(S^3 - L_1).$$

  (C is homotopy ribbon.)
  Surjectivity is easy; injectivity takes some significant 3-manifold topology (Thurston) and group theory (Gerstenhaber–Rothaus).

Ribbon concordance and π_1

Theorem (Gordon 1981)

If $K_0 \preceq K_1$ and $K_1 \preceq K_0$, and $\pi_1(K_1)$ is tranfinitely nilpotent, then $K_0 = K_1$.

- Knots that for which π_1 is transfinitely nilpotent include fibered knots, 2-bridge knots, connected sums and cables of transfinitely nilpotent.
- Nontrivial knots with Alexander polynomial 1 are not transfinitely nilpotent.

Theorem (Silver 1992 + Kochloukova 2006)

If $K_0 \preceq K_1$ and K_1 is fibered, then K_0 is fibered.

Ribbon concordance and polynomial invariants

Theorem (Gordon 1981)

If $L_0 \preceq L_1$*, then* $\deg \Delta(L_0) \leq \deg \Delta(L_1)$*.*

Theorem (Gilmer 1984)

If $L_0 \preceq L_1$*, then* $\Delta(L_0) | \Delta(L_1)$*.*

Theorem (Friedl–Powell 2019)

If there is a (locally flat) homotopy ribbon concordance from L_0 *to* L_1*, then* $\Delta(L_0) | \Delta(L_1)$*.*

Ribbon concordance and polynomial invariants

The analogous divisibility result for the Jones polynomial isn't true, except for...

Theorem (Eisermann 2009)

If L is an n-component ribbon link (i.e. if $O^n \preceq L$), then $V(O^n) | V(L)$.

Link homology theories

- Knot Floer homology and Khovanov homology are each bigraded vector spaces:

$$\widehat{\mathrm{HFK}}(K) = \bigoplus_{a,m\in\mathbb{Z}} \widehat{\mathrm{HFK}}_m(K,a) \qquad \mathrm{Kh}(L) = \bigoplus_{i,j\in\mathbb{Z}} \mathrm{Kh}^{i,j}(L).$$

  $\widehat{\mathrm{HFK}}$ behaves a little bit differently for multi-component links.

- They categorify the Alexander and Jones polynomial, respectively:

$$\Delta(K)(t) = \sum_{a,m} (-1)^m t^a \dim \widehat{\mathrm{HFK}}_m(K,a)$$

$$V(L)(q) = \sum_{i,j} (-1)^i q^j \dim \mathrm{Kh}^{i,j}(L)$$

Link homology theories

- Knot Floer homology detects the genus of a knot (Ozsváth–Szabó):

$$g(K) = \max\{a \mid \widehat{\mathrm{HFK}}_*(K,a) \neq 0\}$$
$$= -\min\{a \mid \widehat{\mathrm{HFK}}_*(K,a) \neq 0\}$$

- ...and whether the knot is fibered (Ozsváth–Szabó, Ghiggini, Ni): K is fibered if $\dim \widehat{\mathrm{HFK}}_*(K, g(K)) = 1$.
- Khovanov homology, like the Jones polynomial, tells us something about the minimal crossing number:

$$\max\{j \mid \mathrm{Kh}^{*,j}(L) \neq 0\} - \min\{j \mid \mathrm{Kh}^{*,j}(L) \neq 0\} \leq 2c(L) + 2,$$

with equality iff L is alternating.

Link homology theories

Both knot Floer homology and Khovanov homology are functorial under (decorated) cobordisms:

- For any (dotted) link cobordism $F \subset S^3 \times [0,1]$ from L_0 to L_1, there's an induced map $\mathrm{Kh}(F)\colon\ \mathrm{Kh}(L_0) \to \mathrm{Kh}(L_1)$, which is homogeneous with respect to the bigrading (of degree determined by the genus), invariant up to isotopy, and functorial under stacking.
  - Khovanov, Jacobsson, Bar-Natan: invariance up to sign, for isotopy in $\mathbb{R}^3 \times [0,1]$.
  - Caprau, Clark–Morrison–Walker: eliminated sign ambiguity.
  - Morrison–Walker–Wedrich: invariance for isotopy in $S^3 \times [0,1]$.
- Juhász, Zemke: Defined similar structure for knot Floer homology — not just for links in S^3 and cobordisms in $S^3 \times [0,1]$, but for arbitrary 3- and 4-manifolds.

Link homology theories and ribbon concordance

Theorem

If C is a (strongly homotopy) ribbon concordance from L_0 to L_1, then C induces a grading-preserving injection of $H(L_0)$ into $H(L_1)$ as a direct summand, where $H(L)$ denotes:

- *Knot Floer homology (Ribbon: Zemke 2019; SHR: Miller–Zemke 2019)*
- *Khovanov homology (Ribbon: L.–Zemke 2019; SHR: Gujral–L. 2020)*
- *Instanton knot homology; Heegaard Floer homology or instanton Floer homology of the branched double cover $\Sigma(L)$ (Lidman–Vela-Vick–Wang 2019)*
- *Khovanov–Rozansky $\mathfrak{sl}(n)$ homology (Ribbon: Kang 2019)*
- *Universal $\mathfrak{sl}(2)$ or $\mathfrak{sl}(3)$ homology; $\mathfrak{sl}(n)$ foam homology (Ribbon: Caprau–González–Lee–Lowrance–Sazdanović–Zhang 2020)*

Corollary (Zemke)

If $L_0 \preceq L_1$, then $g(L_0) \leq g(L_1)$.

Corollary (L.–Zemke)

If $L_0 \preceq L_1$, and L_0 is a non-split alternating link, then $c(L_0) \leq c(L_1)$.

Both of these also apply in the strongly homotopy ribbon setting as well.

Link homology theories and ribbon concordance

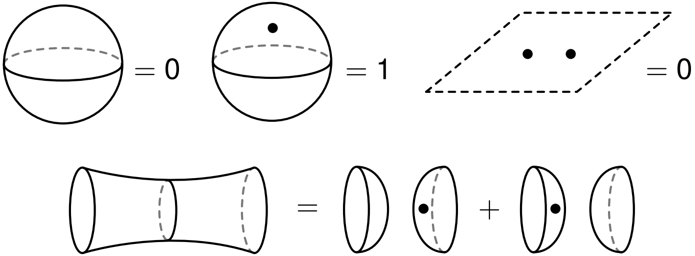
Corollary (Gujral–L. 2020?)

If $L_0 \preceq L_1$, and L_1 is split, then L_0 is split. More precisely, if there is an embedded 2-sphere that separates $L_1^1 \cup \dots \cup L_1^j$ from $L_1^{j+1} \cup \dots L_1^k$, then there is an embedded 2-sphere that separates $L_0^1 \cup \dots \cup L_0^j$ from $L_0^{j+1} \cup \dots L_0^k$.

Several of the above invariants have additional algebraic structure that fully detect splittings; we apply this in conjunction with injectivity.

Khovanov homology and ribbon concordance

The maps on Khovanov homology satisfy several local relations:



Khovanov homology and ribbon concordance

To clarify what these relations mean: Suppose $F \subset S^3 \times [0,1]$ is any cobordism from L_0 to L_1.

- Suppose h is an embedded 3-dimensional 1-handle with ends on F (and otherwise disjoint from F). Let F' be obtained from F by surgery along h, and let $F_1^\bullet$ and $F_2^\bullet$ be obtained by adding a dot to F at either of the feet of h. Then $\mathrm{Kh}(F') = \mathrm{Kh}(F_1^\bullet) + \mathrm{Kh}(F_2^\bullet)$.
- Suppose $S \subset \mathbb{R}^3 \times [0,1]$ is an unknotted 2-sphere that is unlinked from F, and let $S^\bullet$ denote S equipped with a dot. Then $\mathrm{Kh}(F \cup S) = 0$ and $\mathrm{Kh}(F \cup S^\bullet) = \mathrm{Kh}(F)$.
- Rasmussen, Tanaka: The sphere relations also hold for knotted 2-spheres (but still unlinked from F).

Khovanov homology and ribbon concordance

Let C be a ribbon concordance from L_0 to L_1 with r local minima, and let $\overline{C}$ be its mirror, viewed as a concordance from L_1 to L_0. Let $D = C \cup_{L_1} \overline{C}$, and let $I = L_0 \times [0,1]$, both concordances from L_0 to itself.

Lemma (Zemke)

We may find:

- *Unknotted, unlinked* 2*-spheres* $S_1, \dots S_r \subset (S^3 \setminus L_0) \times [0,1]$, *and*
- *Disjointly embedded* 3*-dimensional* 1*-handles* $h_1, \dots, h_r$ *in* $S^3 \times [0,1]$, *where* h_i *joins* I *to* S_i *and is disjoint from* S_j *for* $j \neq i$,

such that D *is isotopic to the surface obtained from* $I \cup S_1 \cup \cdots \cup S_r$ *by embedded surgery along the handles* $h_1, \dots, h_r$.

Khovanov homology and ribbon concordance

- Applying the neck-cutting relation to each of the handles h_i:

$$\begin{aligned} \mathrm{Kh}(D) &= \sum_{\vec{e} \in \{\emptyset, \bullet\}^r} \mathrm{Kh}(I \cup S_1^{e_1} \cup \dots S_r^{e_r}) \\ &= \mathrm{Kh}(I \cup S_1^{\bullet} \cup \dots \cup S_r^{\bullet}) \\ &= \mathrm{Kh}(I) \\ &= \mathrm{id}_{\mathrm{Kh}(L_0)} \end{aligned}$$

- Hence $\mathrm{Kh}(\overline{C}) \circ \mathrm{Kh}(C) = \mathrm{id}_{\mathrm{Kh}(L_0)}$, so $\mathrm{Kh}(C)$ is injective (and left-invertible).
- Caprau–González–Lee–Lowrance–Sazdanović–Zhang: Basically the same proof works for $\mathfrak{sl}(n)$ homology; it's just that the local relations are slightly more complicated.

Khovanov homology and ribbon concordance

- If C is now merely assumed to be strongly homotopy ribbon, Miller and Zemke showed a very similar lemma about the doubled cobordism D, with one catch: the spheres S_i are no longer assumed to be unlinked from I.
- The spheres come from taking the cores of the 2-handles of $S^3 \times [0,1] - \mathrm{nbd}(C)$ with the co-cores of the corresponding handles of $S^3 \times [0,1] - \mathrm{nbd}(\overline{C})$ — possibly multiple pushoffs.

Proposition (Gujral–L., 2020)

The sphere relations for Khovanov homology also hold for linked 2-spheres.

Khovanov homology and splitting of cobordisms

Theorem (Gujral–L., 2020)

Let L_0 and L_1 be links in S^3, with splittings $L_i = L_i^1 \cup \cdots \cup L_i^k$, where the L_i^j are contained in disjoint 3-balls. (The L_i^j may be links, and may even be empty.) Let F be any cobordism from L_0 to L_1 that decomposes as a disjoint union of cobordisms $F^j\colon\ L_0^j \to L_1^j$. Let $\tilde{F}$ be the "split cobordism" consisting of unlinked copies of F^j, each in its own $D^3 \times [0,1]$. Then $\mathrm{Kh}(F) = \pm\,\mathrm{Kh}(\tilde{F})$.

- Proof works by setting up cobordism maps for Batson–Seed's perturbation of Khovanov homology, which is insensitive to crossing changes between different components.