L-spaces, Taut Foliations, Left-Orderability, and Incompressible Tori

Adam Simon Levine

Brandeis University

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- Defined in terms of a chain complex $\widehat{\mathrm{CF}}(\mathcal{H})$ associated to a Heegaard diagram $\mathcal{H}$ for Y:
  - Generators correspond to tuples of intersection points between the two sets of attaching curves.
  - Differential counts holomorphic Whitney disks in the symmetric product — generally a hard analytic problem.

Heegaard Floer homology

- $\widehat{\mathrm{HF}}(Y)$ decomposes as a direct sum of pieces corresponding to spin^c structures on Y:

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Theorem (Ozsváth–Szabó)

If Y is a 3-manifold with $b_1(Y) > 0$, the collection of spin^c structures $\mathfrak{s}$ for which $\widehat{\mathrm{HF}}(Y, \mathfrak{s})$ is nontrivial detects the Thurston norm on $H_2(Y; \mathbb{Z})$. Specifically, for any nonzero $x \in H_2(Y; \mathbb{Z})$,

$$\xi(x) = \max\{\langle c_1(\mathfrak{s}), x\rangle \mid \mathfrak{s} \in \mathrm{Spin}^c(Y), \underline{\mathrm{HF}}(\mathfrak{s}) \neq 0\}.$$

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L-spaces

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- For any rational homology sphere Y and any $\mathfrak{s} \in \mathrm{Spin}^c(Y)$,

$$\dim \widehat{\mathrm{HF}}(Y, \mathfrak{s}) \geq \chi(\widehat{\mathrm{HF}}(Y, \mathfrak{s})) = 1.$$

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$$\dim \widehat{\mathrm{HF}}(Y,\mathfrak{s}) \geq \chi(\widehat{\mathrm{HF}}(Y,\mathfrak{s})) = 1.$$

- Y is called an L-space if equality holds for every spin^c structure, i.e., if

$$\dim \widehat{\mathrm{HF}}(Y) = \left| H^2(Y;\mathbb{Z}) \right|.$$

L-spaces

Examples of L-spaces:

- S^3
- Lens spaces (whence the name)
- All manifolds with finite fundamental group
- Branched double covers of (quasi-)alternating links in S^3

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Question

Can we find a topological characterization (not involving Heegaard Floer homology) of which manifolds are L-spaces?

L-spaces and taut foliations

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Theorem (Ozsváth–Szabó)

If Y is an L-space, then Y does not admit any taut foliation.

Conjecture

If Y is an irreducible rational homology sphere that does not admit any taut foliation, then Y is an L-space.

L-spaces and left-orderability

- A left-ordering on a group G is a total order $<$ such that for any $g, h, k \in G$,

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Conjecture (Boyer–Gordon–Watson, et al.)

Let Y be an irreducible rational homology sphere. Then Y is an L-space if and only if $\pi_1(Y)$ is not left-orderable.

Theorem (L.–Lewallen, arXiv:1110.0563)

If Y is a strong L-space — i.e., if it admits a Heegaard diagram H such that $\dim \widehat{\mathrm{CF}}(\mathcal{H}) = \left|H^2(Y;\mathbb{Z})\right|$ *— then* $\pi_1(Y)$ *is not left-orderable.*

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Theorem (L.–Lewallen, arXiv:1110.0563)

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Theorem (Greene–L.)

For any N, there exist only finitely may strong L-spaces with $|H^2(Y;\mathbb{Z})| = n$.

Conjecture

If Y is an irreducible 3-manifold with $\dim \widehat{\mathrm{HF}}(Y) = 1$, then Y is homeomorphic to either S^3 or the Poincaré homology sphere.

L-space homology spheres

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This is known for all Seifert fibered spaces (Rustamov), graph manifolds (Boileau–Boyer, via taut foliations), and manifolds obtained by Dehn surgery on knots in S^3 (Ozsváth–Szabó).

Incompressible tori

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If Y is an irreducible 3-manifold with $\dim \widehat{\mathrm{HF}}(Y) = 1$, then Y does not contain an incompressible torus.

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By geometrization and Rustamov's work, this would imply that it suffices to look at hyperbolic 3-manifolds for the L-space homology sphere conjecture.

Incompressible tori

If $K_1 \subset Y_1$, $K_2 \subset Y_2$ are knots in homology spheres, let

$$M(K_1, K_2) = (Y_1 \setminus \operatorname{nbd} K_1) \cup_\phi (Y_2 \setminus \operatorname{nbd} K_2)$$

where $\phi \colon \partial(Y_1 \setminus \operatorname{nbd} K_1) \to \partial(Y_2 \setminus \operatorname{nbd} K_2)$ is an orientation-reversing diffeomorphism taking

$$\begin{aligned} \text{meridian of } K_1 &\longrightarrow \text{0-framed longitude of } K_2 \\ \text{0-framed longitude of } K_1 &\longrightarrow \text{meridian of } K_2. \end{aligned}$$

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If Y is a homology sphere and $T \subset Y$ is a separating torus, then $Y \cong Y(K_1, K_2)$ for some $K_1 \subset Y_1$, $K_2 \subset Y_2$, knots in homology spheres, and T is incompressible if and only if K_1 and K_2 are both nontrivial knots.

Incompressible tori

Theorem (Hedden–L., arXiv:1210.7055)

If Y_1 and Y_2 are homology sphere L-spaces, and $K_1 \subset Y_1$ and $K_2 \subset Y_2$ are nontrivial knots, then

$$\dim \widehat{\mathrm{HF}}(M(K_1, K_2)) > 1.$$

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Removing the hypothesis that Y_1 and Y_2 are L-spaces will complete the proof of the incompressible torus conjecture.

Bordered Heegaard Floer homology

Lipshitz, Ozsváth, and Thurston define invariants of 3-manifolds with parametrized boundary:

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Bordered Heegaard Floer homology

Theorem (Lipshitz–Ozsváth–Thurston)

1. $\widehat{\mathrm{CFA}}(M_1)$ *and* $\widehat{\mathrm{CFD}}(M_2)$ *are invariants up to chain homotopy equivalence.*

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Theorem (Lipshitz–Ozsváth–Thurston)

1. $\widehat{\text{CFA}}(M_1)$ *and* $\widehat{\text{CFD}}(M_2)$ *are invariants up to chain homotopy equivalence.*
2. *If* $Y = M_1 \cup_{\phi_2 \circ \phi_1^{-1}} M_2$, *then*

$$\widehat{\text{HF}}(Y) \cong H_*(\widehat{\text{CFA}}(M_1) \,\widetilde{\otimes}_{\mathcal{A}(F)}\, \widehat{\text{CFD}}(M_2)).$$

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- If $K \subset Y$ is a knot in a homology sphere, the bordered invariants of $X_K = Y \setminus \text{nbd}(K)$ are related to to the knot Floer homology of K, $\widehat{\text{HFK}}(Y, K)$, which detects the genus of K.

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- If K_1 and K_2 are nontrivial knots in L-space homology spheres, we can explicitly identify at least two cycles in

$$\widehat{\mathrm{CFA}}(X_{K_1}) \mathbin{\widetilde{\otimes}}_{\mathcal{A}(T^2)} \widehat{\mathrm{CFD}}(X_{K_2})$$

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- Hope to extend this approach for knots in general homology spheres.