Bordered Heegaard Floer Homology and Knot Doubling Operators

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Knot Concordance and Homology Cobordism Workshop
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Slice Knots and Links

Definition

A knot in S^3 is called

- topologically slice if it is the boundary of a locally flatly embedded disk in B^4.
- smoothly slice if it is the boundary of a smoothly embedded disk in B^4.

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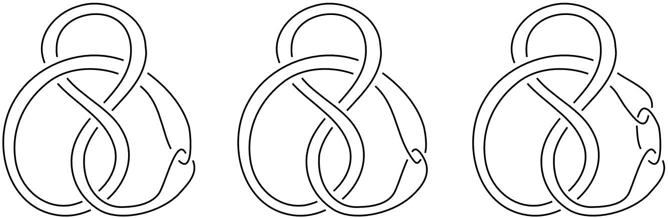
Big question: How do these two notions compare?

Whitehead and Bing Doubling

Given a knot K, the positive Whitehead double, negative Whitehead double, and Bing double are:



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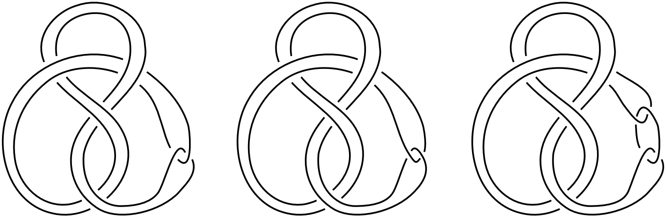

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We consider only untwisted doubles here.

When are Whitehead doubles topologically slice?

Theorem (Freedman)

The Whitehead double (with either sign) of any knot is topologically slice. More generally, if L is a boundary link, then any Whitehead double of L is topologically slice.

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- For two-component links, the answer is yes.
- It is equivalent to the four-dimensional surgery conjecture.
- Most people, including Freedman, think it's not true.

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- Bižaca used this to construct explicit examples of exotic smooth structures on $\mathbb{R}^4$.

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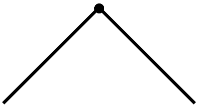
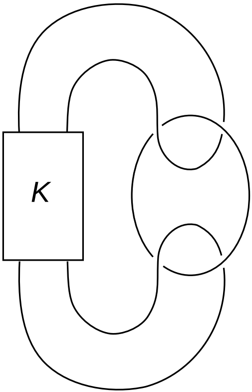
Corollary

If K is any knot with $\tau(K) > 0$ (e.g., any strongly quasipositive knot), then any iterated positive Whitehead double of K is not smoothly slice.

Iterated Bing Doubling

Any binary tree T specifies an iterated Bing double of K, denoted $B_T(K)$.





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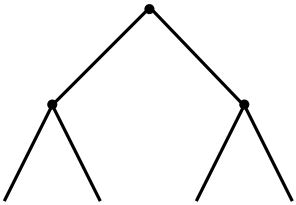

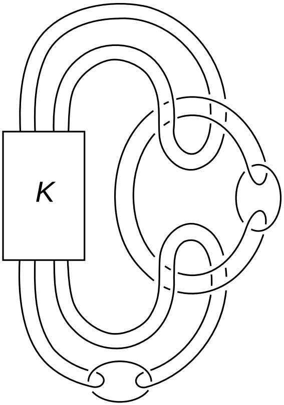

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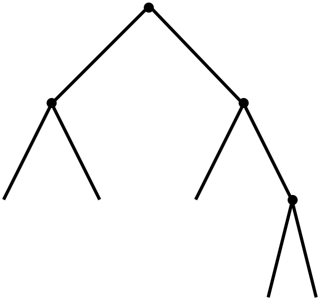
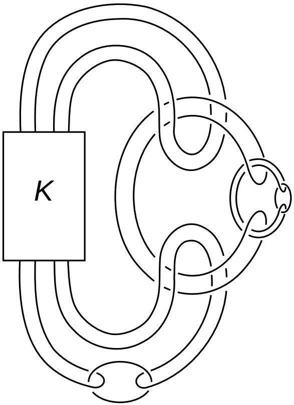
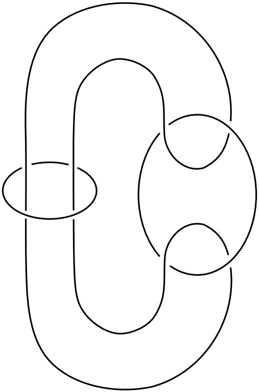
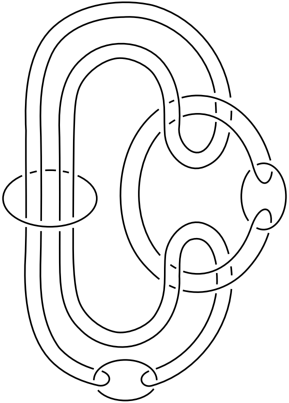




Generalized Borromean Rings

The family of generalized Borromean links consists of all links obtained by taking iterated Bing doubles of the components of the Hopf link.





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1. *Let K be any knot with $\tau(K) > 0$ (e.g., any strongly quasipositive knot), and let T be any binary tree. Then the all-positive Whitehead double of $B_T(K)$ is topologically but not smoothly slice.*

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2. *The all-positive Whitehead double of any generalized Borromean link is not smoothly slice.*

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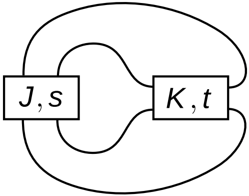
2. *The all-positive Whitehead double of any generalized Borromean link is not smoothly slice.*

It is not known whether the links in (2) are topologically slice.

Doubling operators

- Given knots J, K and integers s, t, define the knot $D_{J,s}(K,t) = D_{K,t}(J,s)$ as the boundary of the plumbing of an s-framed J-annulus and a t-framed K-annulus.



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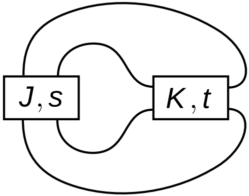

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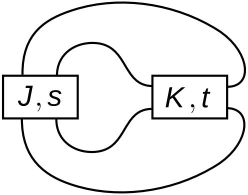

- So $Wh_{\pm}(K) = D_{O,\mp 1}(K,0)$.
- When $t = 0$, we often omit it: $D_{J,s}(K) = D_{J,s}(K,0)$.

Doubling operators

Proposition (Rudolph, Livingston)

If $s \leq TB(J)$ and $t \leq TB(K)$, then $D_{J,s}(K,t)$ is strongly quasipositive, so $\tau(D_{J,s}(K,t)) = 1$.

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$$\tau(D_{J,s}(K,t)) = \begin{cases} 1 & s > 2\tau(J),\ t > 2\tau(K) \\ -1 & s < 2\tau(J),\ t < 2\tau(K) \\ 0 & \text{otherwise.} \end{cases}$$

Covering link calculus

Definition

A link L in a $\mathbb{Z}_2$-homology 3-sphere Y is called $\mathbb{Z}_2$-slice if there exists a $\mathbb{Z}_2$-homology 4-ball X with $\partial X = Y$ such that L bounds disjoint disks in X.

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Theorem (Ozsváth-Szabó)

If $K \subset S^3$ is smoothly $\mathbb{Z}_2$-slice, then $\tau(K) = 0$.

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Lemma

Let L be a link in S^3, and suppose there is an unknotted solid torus $U \subset S^3$ such that $L \cap U$ consists of two components K_1, K_2 embedded as follows: if A_1, A_2 are the components of the untwisted Bing double of the core C of U, then

$$K_1 = D_{P_k,s_k} \circ \cdots \circ D_{P_1,s_1}(A_1), \qquad K_2 = D_{Q_l,t_l} \circ \cdots \circ D_{Q_1,t_1}(A_2).$$

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Let L' be the link obtained from L by replacing K_1 and K_2 by

$$C' = D_{P_k,s_k} \circ \cdots \circ D_{P_1,s_1} \circ D_{R,u}(C), \text{ where}$$

$$(R,u) = \begin{cases} (Q_1 \# Q_1^r,\ 2t_1) & l = 1 \\ (D_{Q_1,t_1} \circ \cdots \circ D_{Q_{l-2},t_{l-2}}(D_{Q_{l-1},t_{l-1}}(Q_l \# Q_l^r, 2t_l)),\ 0) & l > 1. \end{cases}$$

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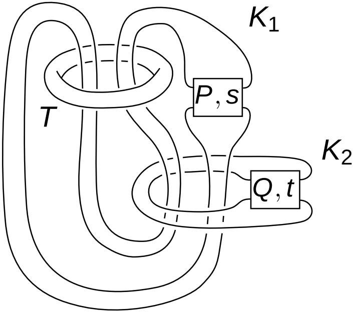
Covering link calculus



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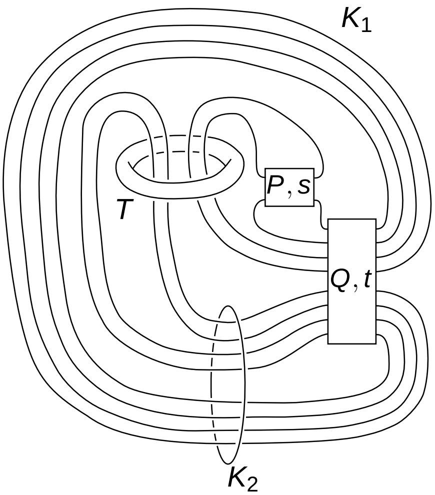

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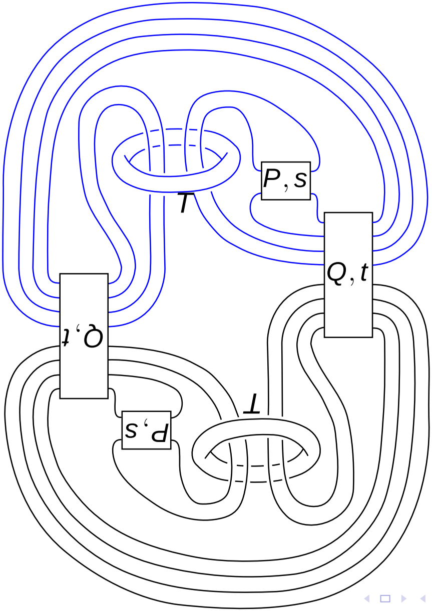

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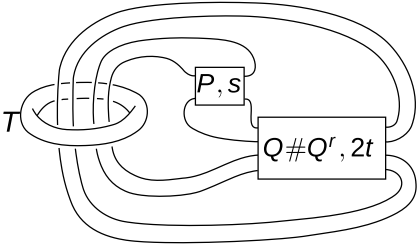

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- Thus, $\tau(D_{P_k,s_k} \circ \cdots \circ D_{P_1,s_1}(K)) = 1$, so $D_{P_k,s_k} \circ \cdots \circ D_{P_1,s_1}(K)$ is not smoothly $\mathbb{Z}_2$-slice, so $Wh_+(B_T(K))$ is not smoothly slice. □

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- If we use a mix of positive and negative Whitehead doubling, this approach fails.

Heegaard Floer Homology

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- If $Y = S^3$, then $\widehat{\mathrm{HF}}(Y) = \mathbb{F}$. $\tau(K)$ is the least filtration of any element of $\widehat{\mathrm{HFK}}(Y, K)$ that survives to the E^∞ page.

Bordered Heegaard Floer homology

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Bordered Heegaard Floer homology

Surface $F \implies$ DG algebra $\mathcal{A}(F)$

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Theorem (Lipshitz–Ozsváth–Thurston)

If $Y = Y_1 \cup_{\phi_1 \circ \phi_2^{-1}} Y_2$, *then*

$$\widehat{\mathrm{CFA}}(Y_1) \,\tilde{\otimes}\, \widehat{\mathrm{CFD}}(Y_2) \simeq \widehat{\mathrm{CF}}(Y).$$

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Moreover, if K *is a nulhomologous knot in either* Y_1 *or* Y_2*, then there is an induced filtration on either* $\widehat{\mathrm{CFA}}(Y_1)$ *or* $\widehat{\mathrm{CFD}}(Y_2)$*, which induces the filtration on* $\widehat{\mathrm{CF}}(Y, K)$*.*

Can also define bimodules. For example:

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There are versions of the gluing theorem for bimodules as well.

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$$\widehat{\mathrm{CF}}(S^3, D_{J,s}(K,t)) \simeq (\widehat{\mathrm{CFAA}}(X) \tilde{\otimes} \widehat{\mathrm{CFD}}(Y_J^s)) \tilde{\otimes} \widehat{\mathrm{CFD}}(Y_K^t).$$

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- We can then follow the spectral sequence from $\widehat{\mathrm{HFK}}(D_{J,s}(K,t))$ to $\widehat{\mathrm{HF}}(S^3)$ carefully to determine $\tau(D_{J,s}(K,t))$.

The torus algebra

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Lipshitz, Ozsváth, and Thurston proved:

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- $\iota_1 \widehat{\mathrm{CFD}}(X_K^t)$ is generated by

$$\{\gamma_1, \dots, \gamma_r\} \cup \bigcup_{j=1}^{n} \{\kappa_1^j, \dots, \kappa_{k_j}^j\} \cup \bigcup_{j=1}^{n} \{\lambda_1^j, \dots, \lambda_{l_j}^j\}.$$

where $r = |2\tau(K) - t|$.

- Vertical stable chains:

$$\xi_{2j} \xrightarrow{\rho_{123}} \kappa^j_1 \xrightarrow{\rho_{23}} \cdots \xrightarrow{\rho_{23}} \kappa^j_{k_j} \xleftarrow{\rho_1} \xi_{2j-1}.$$

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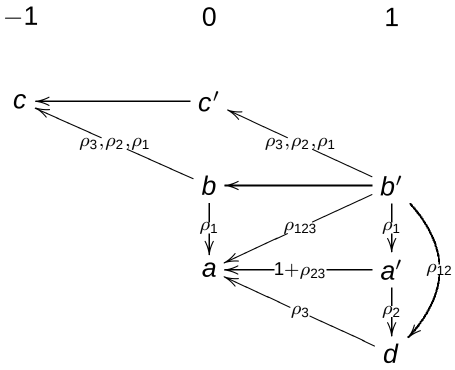
$$\begin{cases} \eta_0 \xrightarrow{\rho_3} \gamma_1 \xrightarrow{\rho_{23}} \cdots \xrightarrow{\rho_{23}} \gamma_r \xleftarrow{\rho_1} \xi_0 & t < 2\tau(K) \\ \xi_0 \xrightarrow{\rho_{12}} \eta_0 & t = 2\tau(K) \\ \xi_0 \xrightarrow{\rho_{123}} \gamma_1 \xrightarrow{\rho_{23}} \cdots \xrightarrow{\rho_{23}} \gamma_r \xrightarrow{\rho_2} \eta_0 & t > 2\tau(K). \end{cases}$$

$\widehat{\mathrm{CFA}}$ for the Whitehead double

Let $Wh \subset S^1 \times D^2$ be the pattern for the positive Whitehead double. Then $\widehat{\mathrm{CFA}}(S^1 \times D^2, Wh)$ has the following form:



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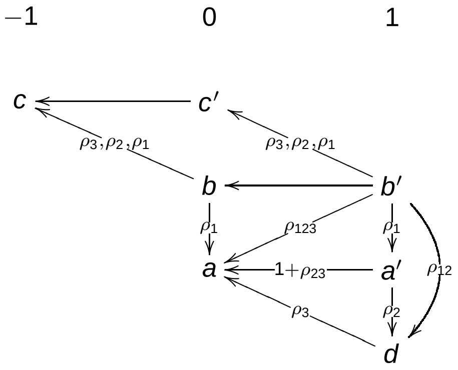

In other words, for instance:

$$m_1(b') = b \qquad m_2(b, \rho_1) = a \qquad m_4(b, \rho_3, \rho_2, \rho_1) = c$$

We split $\widehat{\mathrm{CFA}}(S^1 \times D^2, Wh) \boxtimes \widehat{\mathrm{CFD}}(X_K^t)$ into direct summands according to the horizontal and vertical chains:

$$C^j_{\text{vert}} = \langle b, b' \rangle \boxtimes \langle \xi_{2j-1}, \xi_{2j} \rangle + \langle a, a', c, c' \rangle \boxtimes \left\langle \kappa^j_i \mid 1 \leq i \leq k_j \right\rangle$$

$$C^j_{\text{hor}} = \langle d \rangle \boxtimes \langle \eta_{2j-1}, \eta_{2j} \rangle + \langle a, a', c, c' \rangle \boxtimes \left\langle \lambda^j_i \mid 1 \leq i \leq l_j \right\rangle$$

$$C_{\text{unst}} = \langle b \boxtimes \xi_0, b' \boxtimes \xi_0, d \boxtimes \eta_0 \rangle + \langle a, a', c, c' \rangle \boxtimes \langle \lambda_i \mid 1 \leq i \leq r \rangle .$$

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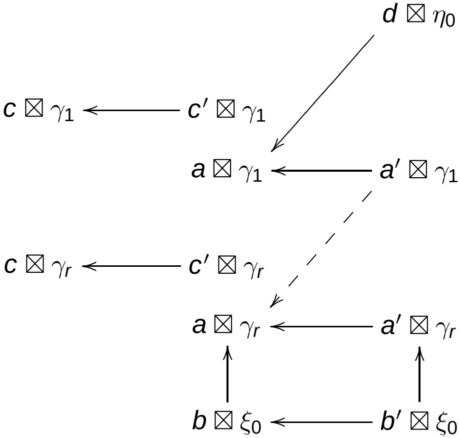
What's special here is that we actually get a direct sum decomposition. Almost. The single $\mathbb{F}$ that survives in homology always comes from C_{unst}.

Proving Hedden's formula for $\tau(Wh_+(K,t))$

In the case where $s < 2\tau(K)$:



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The theorem's proved, the dissertation's done,
But all the work ahead has just begun.