RICCI FLOW SOLITONS IN DIMENSION THREE WITH SO(3)-SYMMETRIES

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ABSTRACT. These are my notes on the Ricci flow solitons in dimension 3 that have an SO(3)-symmetry.

I prove that there is a unique steady example that is complete and of positive curvature.

I prove that there is a 1-parameter family of complete expanding examples with positive curvature.

CONTENTS

1. INTRODUCTION

The goal is to understand the rotationally invariant solutions of the gradient Ricci flow solitons.

A pair (g, f), where g is a Riemannian metric on a manifold M and f is a (smooth) function on M, is said to be a *gradient Ricci flow soliton with expansion constant* λ if it satisfies

$$\mathrm{Ric}(g) = \mathrm{Hess}_g(f) - \lambda\, g, \tag{1.1}$$

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where $\mathrm{Hess}_g(f) = \nabla_g^2 f$ is the Hessian form of f with respect to g.

For such a pair, the Ricci flow heat equation $G_t = -2\,\mathrm{Ric}(G)$ with initial condition $G_0 = g$ has, in the t-interval on which $1{+}2\lambda t > 0$, the solution

$$G = \bigl(1 + 2\lambda t\bigr)\,\Phi_{h(t)}^*(g) \tag{1.2}$$

where $\Phi_\tau : M \to M$ is the (possibly locally defined) time τ flow of $\nabla_g f = \mathrm{grad}_g f$ and the function h is given by

$$h(t) = \begin{cases} -\dfrac{\log(1+2\lambda t)}{2\lambda} &, \quad \lambda \neq 0, \\[2ex] -t &, \quad \lambda = 0. \end{cases} \tag{1.3}$$

When $\lambda < 0$, the soliton is said to be *shrinking*, when $\lambda = 0$, the soliton is said to be *steady*, and when $\lambda > 0$, the soliton is said to be *expanding*.

When (g, f) is a gradient Ricci flow soliton with expansion constant λ, the pair $(\mu g, f + c)$ is a gradient Ricci flow solition with expansion constant λ/μ for any constants c and $\mu > 0$. Thus, one should regard f as being only determined up to a constant and one can always reduce to the cases in which λ equals -1, 0, or 1. (However, this scaling can disturb other normalizations, such as scaling to arrange that the maximum of the sectional curvature be some fixed constant, which is frequently done in the literature.)

Remark 1 (Ambiguities). In the literature, one finds reference to the metric g, by itself, as being a soliton. This allows for some slight ambiguity in the terminology, since it could indeed happen that there could exist distinct functions f and f' and distinct constants λ and λ' such that

$$\mathrm{Ric}(g) = \mathrm{Hess}_g(f) - \lambda\, g = \mathrm{Hess}_g(f') - \lambda'\, g. \tag{1.4}$$

All that is required is that the soliton (g, f) should admit a nonzero function a such that $\mathrm{Hess}_g(a) = \alpha\, g$ for some constant α, for then $(g, f + a)$ will be a soliton with expansion constant $\lambda' = \lambda + \alpha$.

For example, when g is the standard flat metric on $\mathbb{R}^n$, and $f = c + b\cdot x + \frac{1}{2} a\, x\cdot x$ for any constants $a, c \in \mathbb{R}$ and constant vector $b \in \mathbb{R}^n$, one has $\mathrm{Hess}_g(f) = a\, g$. Thus, since $\mathrm{Ric}(g) = 0$ for the flat metric, (g, f) is a gradient Ricci solition with expansion constant a for each such f.

Of course, for the 'generic' metric g, the only solutions of $\mathrm{Hess}_g(a) = \alpha\, g$ for α a constant have $\alpha = 0$ and a is constant, so the only ambiguity is that one can add a constant to f (which one can always do). When g admits a nonconstant function a such that $\mathrm{Hess}_g(a) = 0$, then g is a product metric and these are well-understood. When g admits a function a such that $\mathrm{Hess}_g(a) = \alpha\, g$ where $\alpha \neq 0$, one can reduce to the case $\alpha = 1$ by scaling. These cases can be explictly computed and one finds that the only Ricci soliton among them is the flat metric.

Thus, for all but a handful of metrics, the equation (1.1), when it can be satisfied, determines λ and f (up to an additive constant). This justifies calling g the soliton.

2. The rotational Ansatz

The space of local solutions (g, f) to (1.1) is quite large and not very much is known about its global properties. The simplest case to consider is the case of rotational symmetry, i.e., metrics of the form

$$g = \mathrm{d}t^2 + a(t)^2\,\mathrm{d}\sigma^2 \tag{2.1}$$

where $\mathrm{d}\sigma^2$ is the standard round metric on the unit 2-sphere. Usually, one assumes that the *aperature function* a is nonvanishing, but, in order to take into account the fixed points (if any) of the rotational metric, the function a is allowed to vanish at isolated points t_0 within its interval of definition, as long as $a'(t_0) = \pm 1$ and $a(t_0-\tau) = -a(t_0+\tau)$ for all τ such that $t_0 \pm \tau$ lies in the interval of definition of a.

For such a metric, the Ricci curvature is given by the formula

$$\mathrm{Ric}(g) = -2\frac{a''(t)}{a(t)}\,\mathrm{d}t^2 + \big(1 - a'(t)^2 - a(t)a''(t)\big)\,\mathrm{d}\sigma^2. \tag{2.2}$$

Remark 2 (Sectional Curvatures)*.* The radial sectional curvature of the metric (2.1) is $-\frac{a''(t)}{a(t)}$, while the sectional curvature of planes tangent to the 2-dimensional orbits of the rotation group (i.e., the level sets of t) is $\frac{1-a'(t)^2}{a(t)^2}$.

For any function $f(t)$, the Hessian of f with respect to g is

$$\mathrm{Hess}_g(f) = f''(t)\,\mathrm{d}t^2 + a(t)a'(t)f'(t)\,\mathrm{d}\sigma^2. \tag{2.3}$$

Thus, the equation (1.1) becomes the pair of autonomous ODE

$$\begin{aligned} -2a(t)a''(t) &= a(t)^2\big(f''(t) - \lambda\big),\\ 1 - a'(t)^2 - a(t)a''(t) &= a(t)a'(t)f'(t) - \lambda\, a(t)^2. \end{aligned} \tag{2.4}$$

Note that the function f does not appear explicitly, which is to be expected, since adding a constant to f does not affect the property of (g, f) being a soliton with expansion constant λ.

On any t-interval in which aa' is nonzero, one can solve the second equation of (2.4) for f' and substitute the result into the first equation of (2.4), yielding a (somewhat complicated) third order autonomous ODE for a. However, this equation does not appear to be useful, so it will not be written out explicitly.

Remark 3 (Bianchi conservation law)*.* The usual Bianchi identity for Ricci curvature, when applied to Ricci solitons, shows that any solution of (2.4) satisfies

$$\left(f'(t) + 2\,\frac{a'(t)}{a(t)}\right)^2 - 2\,\frac{1 + a'(t)^2}{a(t)^2} - 2\lambda\, f(t) = C \tag{2.5}$$

for some constant C.

When λ is nonzero, this constant C has no significance, since it is affected by adding a constant to f, which, of course, does not change the metric g.

When $\lambda = 0$ (the steady case), (2.5) turns out to be an important conservation law that allows integration of the equations up to a phase portrait (see below).

2.1. **Analyticity and Duality.** One can write the equations (2.4) in the form

$$\begin{aligned} a(t)^2\, f''(t) &= -2 + 2a'(t)^2 + 2a(t)a'(t)f'(t) - \lambda\, a(t)^2,\\ a(t)\, a''(t) &= 1 - a'(t)^2 - a(t)a'(t)f'(t) + \lambda\, a(t)^2. \end{aligned} \tag{2.6}$$

It follows that, on any t-interval in which a is nonvanishing, the functions a and f are real-analytic.

Recalling that only the derivative of f has any actual significance, one can recast this equation as a 'lower order' system by setting $p(t) = a(t)f'(t)$ and writing the system in the form

$$\begin{aligned} a(t)\,p'(t) &= -2 + 2a'(t)^2 + 3a'(t)p(t) - \lambda\, a(t)^2,\\ a(t)\,a''(t) &= 1 - a'(t)^2 - a'(t)p(t) + \lambda\, a(t)^2. \end{aligned} \tag{2.7}$$

2.1.1. *Fixed points.* A value t_0 for which $a(t_0) = 0$ represents a fixed point of the rotational symmetry. By translation in t, one can assume that $t_0 = 0$.

By the second equation of (2.6), one has $a'(0) = \pm 1$. Thus, there exists an interval $(-T, T)$ about $t = 0$ (where $0 < T \leq \infty$) on which a is an odd function of t whose first derivative is nonvanishing. By reversing t if necessary, it can be assumed that $a'(0) = 1$, so that $a'(t) > 0$ for all $|t| < T$. Consequently, there is a function h, positive and smooth on some interval $(-\varepsilon, A)$ where $A, \epsilon > 0$ and satisfying $h(0) = 1$, such that

$$g = \frac{\mathrm{d}a^2}{h(a^2)} + a^2\,\mathrm{d}\sigma^2. \tag{2.8}$$

Setting $r = a^2$, one finds that the ODE for h that makes g into a soliton with expansion constant λ is

$$2r^2h(r)h''(r) = h(r)\big(h(r) - 1\big) + rh'(r)\big(rh'(r) - \lambda r - 1\big), \qquad r \geq 0. \tag{2.9}$$

The corresponding function f can be found by quadrature via the equation

$$\mathrm{d}f = \frac{a^2\big(h'(a^2) - \lambda\big) + h(a^2) - 1}{ah(a^2)}\,\mathrm{d}a = \frac{r\big(h'(r) - \lambda\big) + h(r) - 1}{2rh(r)}\,\mathrm{d}r. \tag{2.10}$$

Because $h(0) = 1$ and h is smooth and positive on the interval of definition, these latter two expressions are smooth in a^2 and r, respectively. Moreover, since h is real-analytic on $(0, A)$, it follows that f must also be real-analytic as a function of r on $(0, A)$. The issue is whether h is real-analytic at $r = 0$.

Now, (2.9) has a singular point at $r = 0$.[1] However, there are real-analytic solutions to (2.9) defined on a neighborhood of $r = 0$.

Proposition 1 (Singular solutions)**.** *For any constant h_1, the equation (2.9) has a unique real-analytic solution about $r = 0$ satisfying $h(0) = 1$ and $h'(0) = h_1$.*

Moreover, any solution of (2.9) that is defined and C^1 on an interval of the form $0 \leq r < \epsilon$ or $-\epsilon < r \leq 0$ and that satisfies $h(0) = 1$ is real-analytic at $r = 0$.

Proof. Straightforward calculation shows that the formal power series

$$h(r) = 1 + h_1\, r + h_2\, r^2 + \cdots \tag{2.11}$$

satisfies (2.9) if and only if, for all $k \geq 2$,

$$(2k{+}1)(k{-}1)\, h_k + \lambda(k{-}1)\, h_{k-1} + \sum_{j=1}^{k-1}(k^2{-}3kj{+}3j^2{-}k{-}1)\, h_j h_{k-j} = 0. \tag{2.12}$$

Thus, specifying h_1 determines h_k uniquely for $k \geq 2$. From the form of (2.12), it follows that for $k \geq 2$, there exist polynomials $H_k(u, v)$, homogeneous of total degree k in u and v, such that the solution of (2.12) is given by $h_k = H_k(h_1, \lambda)$. For example, $H_2(u, v) = \frac{1}{5}u(2u - v)$, etc. For consistency of notation, set $H_1(u, v) = u$.

[1] The order of the singularity can be reduced by one by making the substitution $h(r) = 1 - r\sigma(r)$, but it cannot be removed entirely.

It must now be shown that the formal power series thus determined has a positive radius of convergence. This is most easily done by the method of majorants. Observe that, when $1 \le j < k$, the inequality

$$\left|k^2 - 3kj + 3j^2 - k - 1\right| \le \tfrac{1}{2}(2k+1)(k-1) \tag{2.13}$$

holds. It follows that the sequence h_k defined by (2.12) satisfies $|h_k| \le a_k$ when a_k is the sequence defined by $a_1 = |h_1|$ and

$$a_k = |\lambda|\, a_{k-1} + \tfrac{1}{2}\sum_{j=1}^{k-1} a_j a_{k-j} \tag{2.14}$$

for $k \ge 2$. This latter sequence is the sequence of coefficients of the analytic function $a(r) = a_1 r + a_2 r^2 + \cdots$ that satisfies the equation

$$a(r) - |\lambda|\, r a(r) - \tfrac{1}{2}\, a(r)^2 = a_1\, r. \tag{2.15}$$

Explicitly, $a(r) = 1 - |\lambda|\, r - \sqrt{(1-|\lambda|\, r)^2 - 2a_1\, r}$. In particular the Taylor series for $a(r)$ has a positive radius of convergence about $r = 0$. Since $|h_k| \le a_k$ for all $k \ge 1$, the series (2.11) also has a positive radius of convergence.

Finally, suppose that h is a solution of (2.9) that is defined and C^1 on an interval $0 \le r < \epsilon$ or $-\epsilon < r \le 0$ and such that $h(0) = 1$. Writing (2.9) in the form

$$2h(r)\big(r(rh'(r))' - rh'(r)\big) = h(r)\big(h(r) - 1\big) + rh'(r)\big(rh'(r) - \lambda r - 1\big) \tag{2.16}$$

note that the curve $(x, y, z) = \big(r,\, h(r){-}1,\, rh'(r)\big)$ for $r \ne 0$ is, up to reparametrization, an integral curve of the vector field

$$(\dot{x}, \dot{y}, \dot{z}) = \big(2(1{+}y)x,\, 2(1{+}y)z,\, (y{+}2z)(1{+}y) + z(z - \lambda x - 1)\big). \tag{2.17}$$

that α-limits to $(x, y, z) = (0, 0, 0)$. The linearization of this vector field at $(0, 0, 0)$ is

$$(\dot{x}, \dot{y}, \dot{z}) = \big(2x,\, 2z,\, y{+}z\big). \tag{2.18}$$

and its eigenvalues are 2, 2, and -1. Thus, the curve $\big(r,\, h(r){-}1,\, rh'(r)\big)$ must lie on the (2-dimensional) unstable manifold at $(0, 0, 0)$ of the vector field (2.17) and its tangent space at $(0, 0, 0)$ is the subspace of vectors of the form (a, b, b).

Now, the analytic solutions of (2.9) already found above give curves of the form

$$(x, y, z) = \left(r, h_1 r + \sum_{k=2}^{\infty} H_k(h_1, \lambda) r^k, h_1 r + \sum_{k=2}^{\infty} k H_k(h_1, \lambda) r^k\right) \tag{2.19}$$

where h_1 is an arbitrary constant. These curves lie in the 2-dimensional real-analytic surface parametrized in the form

$$X(r, s) = \left(r, s + \sum_{k=2}^{\infty} H_k(s, \lambda r), s + \sum_{k=2}^{\infty} k H_k(s, \lambda r)\right). \tag{2.20}$$

and, are, in fact, the curves of the form $X(r, h_1 r)$. Thus, for r and s small, $X(r, s)$ must parametrize the 2-dimensional unstable manifold of the vector field (2.17). In particular, these curves $X(r, h_1 r)$ and $X(0, s)$ must represent all of the integral curves of (2.17) that α-limit to $(0, 0, 0)$. It follows that the given solution $h(r)$ must agree with one of these curves for r sufficiently small and hence must be real-analytic at $r = 0$, as desired. □

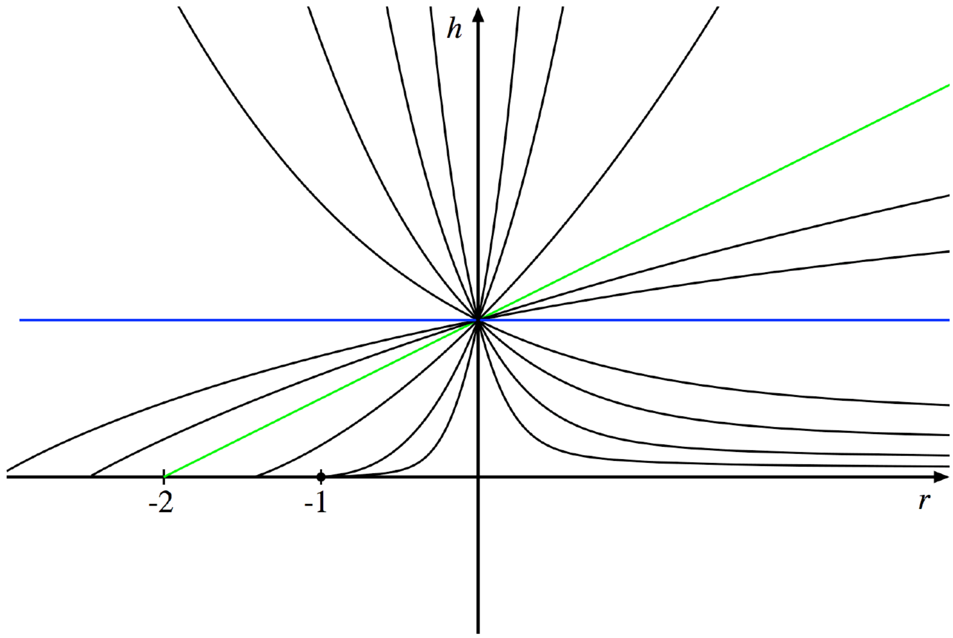

FIGURE 1. Solutions of the equation (2.9) with $\lambda = 1$ that satisfy $h(0) = 1$. The blue line represents the flat solution and the green line represents the constant curvature solution. (The negative values of r are useful for understanding the case $\lambda = -1$. See Remark 7.)

Remark 4 (The h-series near $r = 0$). Explicitly, the first few terms of the solution of (2.9) that satisfies $h(0) = 1$ and $h'(0) = h_1$ are given by

$$h(r) = 1 + h_1\, r + \frac{h_1(2h_1-\lambda)}{5}\, r^2 + \frac{h_1(2h_1-\lambda)(h_1-\lambda)}{35}\, r^3 - \frac{h_1(2h_1-\lambda)(6{h_1}^2+2\lambda h_1-15\lambda^2)}{4725}\, r^4 + \cdots \tag{2.21}$$

For a graphical depiction of these solutions when $\lambda = 1$, see Figure 1.

Proposition 1 yields the existence of a 1-parameter family of solitons for each expansion constant λ in a neighborhood of any fixed point of the rotational action. Of interest will be the behavior of these solutions as r increases, which will be taken up below.

It is also worthwhile to describe the behavior of the nonsmooth solutions defined on an interval $0 < r < \epsilon$.

Proposition 2. *There exists a function G that is real-analytic on a neighborhood of $(0,0,0) \in \mathbb{R}^3$ such that, for any solution h of (2.9) that is defined on the interval $0 < r < \epsilon$ but that does not extend real-analytically to $r = 0$, there exist constants $c_0 \neq 0$ and c_1 and a δ satisfying $0 < \delta \leq \epsilon$ so that*

$$h(r) = \frac{c_0}{r^{\sqrt{2}-1}}\, G\left(\lambda r,\ \lambda\, r\ln(r) + c_1 r,\ \frac{r^{\sqrt{2}-1}}{c_0}\right) \qquad \textit{for} \quad 0 < r < \delta. \tag{2.22}$$

Conversely, for any constants $c_0 \neq 0$ and c_1, there is an $\epsilon > 0$ such that (2.22) defines a solution to (2.9) on the interval $0 < r < \epsilon$. This solution will be positive near $r = 0$ if and only if $c_0 > 0$.

The function G has a convergent power series expansion of the form

$$G(u,v,w) = \sum_{i+j\leq k} a_{ijk}\, u^i v^j w^k \tag{2.23}$$

and is uniquely determined by the additional condition that

$$G(u,v,w) = 1 + \left(\frac{\sqrt{2}}{2} + \frac{2-\sqrt{2}}{4} v\right) w + O(w^2). \tag{2.24}$$

Proof. (Sketch) One looks for solutions of the form

$$h(r) = \frac{c_0}{r^\alpha}\left(1 + \sum_{j=1}^{\infty} h_j(r)\left(\frac{r^\alpha}{c_0}\right)^j\right) \tag{2.25}$$

where $c_0 \neq 0$ and $\alpha > 0$ are constants and $h_j(r)$ is a polynomial of degree at most j in r and $r \ln r$.

Substituting (2.25) into (2.9) and collecting like powers of r^α, one finds, by examining the lowest power of r^α, that $\alpha > 0$ must satisfy $\alpha^2 - 2\alpha - 1 = 0$, i.e., $\alpha = \sqrt{2} - 1$. Examining the next lowest power of r^α, one then obtains

$$h_1(r) = \frac{\sqrt{2}}{2} + \frac{2-\sqrt{2}}{4}(\lambda r \ln r + c_1 r) \tag{2.26}$$

for some constant c_1, and these are the only possibilities for linear combinations of 1, λr and $\lambda r \ln r + c_1 r$. The successive higher powers of r^α then recursively define, for $j \geq 2$, unique functions $h_j(r)$ that can be written in the form $h_j(r) = H_j(\lambda r,\, \lambda r \ln r + c_1 r)$ where $H_j(x, y)$ is a polynomial of degree at most j in x and y whose coefficients belong to the field $\mathbb{Q}(\sqrt{2})$. The series

$$G(u,v,w) = 1 + \left(\frac{\sqrt{2}}{2} + \frac{2-\sqrt{2}}{4} v\right) w + \sum_{j=2}^{\infty} H_j(u,v) w^j \tag{2.27}$$

can now be proved to have a positive radius of convergence by the method of majorants, using the recursive definition of the H_j.

Once G has been shown to exist, it is not difficult to show that any solution of (2.9) that is defined on an interval $0 < r < \epsilon$ and does not extend smoothly to $r = 0$ must be of the form claimed for the appropriate constants c_0 and c_1.

Details will be supplied later. □

Remark 5 (Sectional curvatures). In terms of the representation (2.8), one finds that the radial sectional curvature is $-h'(r)$ and the orbital sectional curvature is $\big(1 - h(r)\big)/r$. Of course, these are equal at $r = 0$ for solutions h that extend smoothly to $r = 0$.

On the other hand, for the solutions h described in Proposition 2, the function $-h'(r)$ goes to $+\infty$ as $r \to 0^+$ while $\big(1 - h(r)\big)/r$ goes to $-\infty$ as $r \to 0^+$. Meanwhile, for the corresponding metric $g = \mathrm{d}t^2 + a(t)^2\, \mathrm{d}\sigma^2$, one has $r = a^2$ and hence

$$\mathrm{d}t = \frac{\mathrm{d}a}{\sqrt{h(a^2)}} = \frac{a^{\sqrt{2}-1}\,\mathrm{d}a}{\sqrt{c_0\,G\left(\lambda a^2,\ \lambda\,a^2\ln(a^2)+c_1a^2,\ a^{2\sqrt{2}-2}/c_0\right)}}. \tag{2.28}$$

Thus, the metric completion to $r=0$ of such a soliton simply adds a fixed point at finite distance, but this point is singular.

Remark 6 (Constant curvature solutions). Note that $h(r)\equiv 1$ and $h(r)\equiv 1+\frac{1}{2}\lambda r$ are explicit solutions of (2.9), defined for all r. The first of these represents the flat metric on $\mathbb{R}^3$, i.e., $g=\mathrm{d}a^2+a^2\,\mathrm{d}\sigma^2$, where $f=-\frac{1}{2}\lambda r=-\frac{1}{2}\lambda a^2$. The second solution, distinct from the first as long as $\lambda\neq 0$, is a non-flat metric,

$$g=\frac{\mathrm{d}a^2}{1+\frac{1}{2}\lambda a^2}+a^2\,\mathrm{d}\sigma^2, \qquad f=0. \tag{2.29}$$

Of course, this is a representation of the metric of constant sectional curvature $-\frac{1}{2}\lambda$. When $\lambda>0$, this formula describes a complete metric. When $\lambda<0$, this formula does not give a complete metric, but represents the metric on a hemisphere of the complete metric on S^3.

Remark 7 (Duality and Scaling). There is a *principle of duality* relating expanding solitons with shrinking solitons: If h satisfies (2.9) with $h(0)=1$ and is positive on an interval (L,M) where $-\infty\leq L<0<M\leq\infty$, then the function k defined on the interval $(-M,-L)$ by the relation $k(r)=h(-r)$ satisfies the equation

$$2r^2k(r)k''(r)=k(r)\bigl(k(r)-1\bigr)+rk'(r)\bigl(rk'(r)+\lambda\,r-1\bigr), \tag{2.30}$$

which is the same as (2.9) with λ replaced by $-\lambda$. Thus, in some sense, the expanding solitons near a fixed point of the rotation are analytic continuations of contracting solitons near a fixed point of the rotation, and *vice versa*.

More generally, if h satisfies (2.9) and is positive on an interval (L,M), then for any constant $\mu\neq 0$, the function ℓ defined on the interval $\mu^{-1}\cdot(L,M)$ by the relation $\ell(r)=h(\mu r)$ satisfies the equation

$$2r^2\ell(r)\ell''(r)=\ell(r)\bigl(\ell(r)-1\bigr)+r\ell'(r)\bigl(r\ell'(r)-\mu\lambda\,r-1\bigr), \tag{2.31}$$

which is the same as (2.9) with λ replaced by $\mu\lambda$. For $\mu>0$, this corresponds to the effect of scaling the metric g.

2.2. **The *xyz*-curve.** Another way of expressing the soliton system is to consider the quantities

$$x(t)=a(t)^2 \qquad y(t)=a'(t)^2 \qquad z(t)=a(t)a''(t). \tag{2.32}$$

All of these quantities are 'geometric': x is the square of the aperature, $(1-y)/x$ is the 2-sphere orbit sectional curvature, and $-z/x$ is the radial sectional curvature. For any function a, the xyz-space curve (2.32) will satisfy the relation

$$x\,\mathrm{d}y-z\,\mathrm{d}x=0. \tag{2.33}$$

The equation (2.4) induces a relation on a that can be expressed in terms of the xyz-space curve as

$$2xy\,\mathrm{d}z-\bigl((y+z)(y+z-1)-\lambda xz\bigr)\,\mathrm{d}x=0. \tag{2.34}$$

For any soliton, the xyz-curve will be an immersion except at points $t=t_0$ where $a(t_0)=a''(t_0)=0$ or $a'(t_0)=a'''(t_0)=0$. (Remember that one cannot have $a(t_0) = a'(t_0) = 0$ at any point t_0 lest the formula (2.1) fail to define a smooth metric at $t = t_0$.) In order to study these points, it suffices to take $t_0 = 0$, which will be done from now on.

As has already been seen, a solution $\bigl(a(t), f(t)\bigr)$ of (2.4) with $a(0) = 0$ is analytic, with a an odd function of t and f an even function of t. Consequently, the functions $x(t)$, $y(t)$, and $z(t)$ defined by (2.32) are analytic functions of t^2, satisfying $\bigl(x(0), y(0), z(0)\bigr) = (0, 1, 0)$. Moreover, writing $x(t) = r(t^2)$, one knows that $r'(0) = 1$, so that one can parametrize the image curve in the form

$$\bigl(x(t), y(t), z(t)\bigr) = \bigl(r, h(r), rh'(r)\bigr), \tag{2.35}$$

where h is a solution of (2.9) that satisfies $h(0) = 1$. Consequently, the image of the xyz-curve near the 'singular' point is a half-closed interval in a smooth integral curve of the two 1-forms (2.33) and (2.34).

Now consider the other kind of 'cusp': A solution of (2.4) with $a'(0) = a'''(0) = 0$. Of course, $a(0) \neq 0$ and a short calculation shows that the equations (2.4), together with the conditions $a'(0) = a'''(0) = 0$ imply that $f'(0) = 0$. It is then easy to see that the unique power series solution to (2.4) with $a(0) = a_0$, $a'(0) = 0$, $f(0) = f_0$, and $f'(0) = 0$ is even in t (i.e., is a series in t^2) and, in fact, takes the form

$$\begin{aligned} a(t) &= a_0 + \frac{1 + \lambda {a_0}^2}{2a_0}\, t^2 + \cdots, \\ f(t) &= f_0 - \frac{2 + \lambda {a_0}^2}{2{a_0}^2}\, t^2 + \cdots. \end{aligned} \tag{2.36}$$

The evenness of a and f implies that the soliton (g, f) is invariant under the involution $t \mapsto -t$. Thus, the metric g has a reflectional symmetry across the 2-sphere $t = 0$ and this sphere is fixed by the flow of the vector field $\operatorname{grad}_g f$.

If $1 + \lambda {a_0}^2 = 0$, the solution is simply $\bigl(a(t), f(t)\bigr) = (a_0, f_0)$, which describes the product metric on the cylinder. The xyz-curve is constant, with image equal to $({a_0}^2, 0, 0)$. (This case only arises when $\lambda < 0$, i.e., for shrinking solitions.)

If $1 + \lambda {a_0}^2 \neq 0$, then the xyz-curve is an embedding (near $t = 0$) of the half-interval $t \geq 0$. In particular, the image must lie in a smooth curve of the form $(x, y, z) = \bigl(r, h(r), rh'(r)\bigr)$ where h is an analytic solution of (2.9) defined on a neighborhood of $r = {a_0}^2$ and satisfying the initial conditions $h({a_0}^2) = 0$ and $h'({a_0}^2) = (1 + \lambda {a_0}^2)/{a_0}^2 \neq 0$. There is a unique such (necessarily convergent) power series solution and it has the form

$$h(r) = \frac{1 + \lambda {a_0}^2}{{a_0}^2}\,(r - {a_0}^2) + \frac{2 + \lambda {a_0}^2}{9{a_0}^6}\,(r - {a_0}^2)^3 + \cdots. \tag{2.37}$$

3. The steady cases

In the steady case, i.e., when $\lambda = 0$, the equations simplify somewhat and one can get a more detailed picture of the solutions.

3.1. **A rational first integral.** In the first place, the differential equations (2.33) and (2.34) simplify to

$$x\,\mathrm{d}y - z\,\mathrm{d}x = 2xy\,\mathrm{d}z - (y + z)(y + z - 1)\,\mathrm{d}x = 0. \tag{3.1}$$

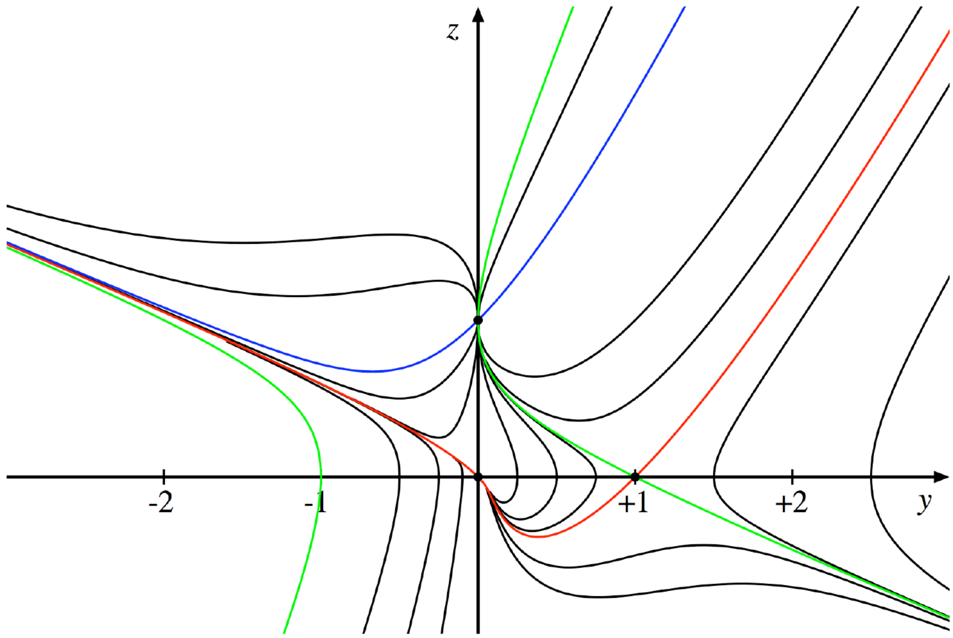

FIGURE 2. Integral curves of the relation (3.3). The red curve, the critical separatrix that provides the complete steady soliton, is smooth but not analytic at the origin. The green hyperbola is the union of the curves on which $C = 0$ (see (3.2).)

Moreover, the relation (2.5) then yields the following rational function C as a first integral of the 1-forms (3.1)

$$C = \frac{(1 - 2z - y^2 - 2yz + z^2)}{xy}, \tag{3.2}$$

at least, away from the planes $x = 0$ and $y = 0$. On an integral curve for which $C \neq 0$, one can use this to express x explicitly as a function of y and z.

To go further, one can eliminate x from the relations (3.1) and get

$$2yz\,\mathrm{d}z - (y + z)(y + z - 1)\,\mathrm{d}y = 0. \tag{3.3}$$

This 1-form has three singular points: $(y, z) = (0, 0)$, $(0, 1)$, and $(1, 0)$. Moreover, the hyperbola $y^2 + 2yz - z^2 + 2z - 1 = 0$, which passes through the second and third of these points, is the union of the integral curves for which $C = 0$. See Figure 2, which will be justified below.

3.2. **A complete, nonnegatively curved soliton.** For the purpose of studying complete steady solitons with nonnegative sectional curvatures, one is interested only in the integral curves of (3.3) that lie in the half-strip described by the inequalities $0 \leq y \leq 1$ and $z \leq 0$, since the integral curves of interest satisfy $x \geq 0$ and $y \geq 0$ and the sectional curvatures of the corresponding solution are $(1-y)/x$ and $-z/x$. Thus, the only integral curve that could provide such a solution is the portion of the red separatrix in Figure 2 that joins the two fixed points $(0, 0)$

and $(1,0)$. The first task is to establish properties of this separatrix (including its existence).

Proposition 3 (The red separatrix). *There is a unique solution to the* ODE

$$2yz\,\frac{\mathrm{d}z}{\mathrm{d}y} = (y+z)(y+z-1) \tag{3.4}$$

that satisfies $z(0) = z(1) = 0$. It is real-analytic everywhere except at $y = 0$, where it is smooth and has a (nonconvergent) Taylor series expansion of the form

$$z(y) = -y - 2y^2 - 12y^3 - \cdots . \tag{3.5}$$

Corollary 1 (A complete steady soliton). *Up to constant multiples, there is a unique, complete, smooth, rotationally invariant steady soliton with positive curvature.*

Proof. This follows by considering the integral curve of (3.3) that joins the two singular points $(0,0)$ and $(1,0)$, as it is the only integral curve that satisfies $1-y > 0$ and $-z > 0$, so that, setting

$$x = \frac{(1 - 2z - y^2 - 2yz + z^2)}{y} > 0 \tag{3.6}$$

(i.e., setting $C = 1$ in (3.2)), the metric g defined by

$$g = \frac{\mathrm{d}x^2}{4xy} + x\,\mathrm{d}\sigma^2 = \frac{\big(\mathrm{d}(a(t)^2)\big)^2}{4a(t)^2a'(t)^2} + a(t)^2\,\mathrm{d}\sigma^2 = \mathrm{d}t^2 + a(t)^2\,\mathrm{d}\sigma^2 \tag{3.7}$$

will have $x \geq 0$ and will have positive sectional curvatures $(1-y)/x$ and $-z/x$. The issue is whether even this metric is complete.

By Proposition 3, this curve is a graph of the form $z = f(y)$ where f is analytic for $0 < y \leq 1$ and is smooth at $y = 0$ with a Taylor series expansion of the form

$$f(y) = -y - 2\,y^2 - 12\,y^3 - \cdots . \tag{3.8}$$

It is also known (see below) that f has a (convergent) Taylor expansion at $y = 1$ of the form

$$f(y) = (y-1) + \tfrac{2}{5}(y-1)^2 - \tfrac{36}{175}(y-1)^3 + \cdots . \tag{3.9}$$

The Taylor expansion of $x(y)$ at $y = 1$ now shows that the metric completes smoothly by adding a point for $y = 1$ (where $x(y)$ vanishes).

Since

$$x(y) = \frac{(1 - 2f(y) - y^2 - 2yf(y) + f(y)^2)}{y} > 0, \tag{3.10}$$

the radial distance t from the center (i.e., $y = 1$) is given by

$$t = \int_y^1 \frac{-x'(\eta)\mathrm{d}\eta}{2\sqrt{\eta\, x(\eta)}}. \tag{3.11}$$

Using the formula for $x(y)$, one finds that t has the expansion

$$t = \frac{1}{2y} - \frac{\ln y}{2} + \tau(y) \tag{3.12}$$

where τ is a smooth function of y near $y = 0$. Thus, the metric g is complete in the direction $y \to 0$. ∎

Remark 8 (Asymptotics of sectional curvature). One also has the expansions

$$\frac{(1-y)}{x(y)} = y - 3y^2 + O(y^3) \qquad \text{and} \qquad \frac{-z(y)}{x(y)} = y^2 + 6y^4 + O(y^5). \tag{3.13}$$

Thus, by (3.12), in an asymptotic expansion in terms of t, the orbital sectional curvature has leading order term $1/(2t)$ while the radial sectional curvature has leading order term $1/(4t^2)$.

Proof. (of Proposition 3) It is easily established that the only formal power series of the form

$$z = -c_1\, y - c_2\, y^2 - \cdots = -\sum_{j=1}^{\infty} c_j\, y^j \tag{3.14}$$

that satisfies (3.4) is given by $c_1 = 1$, $c_2 = 2$, $c_3 = 12$ and the recursion formula

$$c_{j+2} = 2(j+2)c_{j+1} + \sum_{p=2}^{j}(2p-1)c_p c_{j+2-p}\,, \qquad \text{for } j \geq 2. \tag{3.15}$$

This recursion implies $c_j \geq 2^{j-2} j!$ for $j \geq 1$, so this power series solution has a radius of convergence equal to 0. Thus, there is no real-analytic solution of (3.4) that satisfies $z(0) = 0$.

Now, consider the vector field defined by the equations

$$\begin{aligned} \dot{y} &= 2yz, \\ \dot{z} &= (y+z)(y+z-1) \end{aligned} \tag{3.16}$$

In the interior of the wedge W defined by the inequalities $z + y \leq 0 \leq y$, the flow of this vector field has y monotone decreasing and z monotone increasing and there are no fixed points in this wedge other than at $(y,z) = (0,0)$. Consequently, all integral curves of this vector field in W tend to $(0,0)$. Moreover, it is easy to show that any integral curve of this vector field in the half-plane $y \geq 0$ that tends to $(0,0)$ (in either forward or backwards time) either satisfies $y \equiv 0$ or else enters W at some point in forward time.

The next step is to show that there is at least one smooth solution $z = z_0(y)$ on an interval $|y| < \epsilon$ of (3.4) that has its Taylor series at $y = 0$ of the form (3.14) with the c_j defined by $c_1 = 1$, $c_2 = 2$, $c_3 = 12$ and the recursion (3.15). To see this, first make the substitution $z = -y\sqrt{1+w}$, where one assumes that $w > -1$ and the positive square root is intended, so that this can parametrize the two quadrants where $yz < 0$. One finds that the equation (3.4) becomes

$$y^2 \frac{\mathrm{d}w}{\mathrm{d}y} = (1-2y)\sqrt{1+w} - yw - 1 = F(y,w), \tag{3.17}$$

where F is an analytic function on the half-plane $w > -1$. By the above calculations, (3.17) has a unique (non-convergent) power series solution of the form $w = 4\,y + 28\,y^2 + \cdots$. Since (3.17) is of the form treated in [1], those results[2] show that there is at least one smooth solution $w = w_0(y)$ whose Taylor series at $y = 0$ is the formal power series solution. Tracing this back, i.e., setting $z_0(y) = -y\sqrt{1+w_0(y)}$,

[2]See Theorems 12.1 and 14.1 of [1]. However, note that one must make the change of independent variable $y = 1/x$ to align with the notation there. Especially, note that these theorems imply that there is a smooth solution of (3.17) with $w(0) = 0$ defined on an interval $|y| < \epsilon$ for some $\epsilon > 0$ and not merely for $0 \leq y \leq \epsilon$.

one sees that (3.4) has a smooth solution $z_0(y)$ whose Taylor series at $y = 0$ is necessarily equal to the (unique) formal power series solution (3.14).

It remains to understand the behavior of the general solution u of (3.4) that satisfies $\lim_{y\to 0^+} u(y) = 0$. Make the substitution $z = z_0(y)+z_1(y)\,v$, where $z_1(y) \equiv 0$ for $y \le 0$ while $z_1(y) = \mathrm{e}^{-1/(2y)}/y^2$ for $y > 0$. The function z_1 is smooth on the real line and, using the Taylor series for z_0, one sees that (3.4) pulls back under this substitution to be of the form

$$\frac{\mathrm{d}v}{\mathrm{d}y} = v\frac{\big(-8y^4 + y^5 f_2(y) + z_0(y)z_1(y)(5yv-1)\big)}{2y^2 z_0(y)\big(z_0(y) + z_1(y)\,v\big)} \tag{3.18}$$

for some function f_2 that is smooth on a neighborhood of $y = 0$. Using the facts that the function $f_0(y) = z_0(y)/y$ is smooth and nonvanishing near $y = 0$ and the function $f_1(y) = z_1(y)/y^3$ is smooth near $y = 0$, one sees that this equation can be written as

$$\frac{\mathrm{d}v}{\mathrm{d}y} = v\frac{\big(-8 + yf_2(y) + f_0(y)f_1(y)(5yv-1)\big)}{2f_0(y)\big(f_0(y) + y^2 f_1(y)\,v\big)}, \tag{3.19}$$

where the right hand side is smooth along the line $y = 0$. Consequently, every solution u of (3.4) that satisfies $\lim_{y\to 0^+} u(y) = 0$ is of the form

$$u(y) = z_0(y) + \frac{\mathrm{e}^{-1/(2y)}}{y^2}v(y), \qquad y \ge 0. \tag{3.20}$$

where v is a solution of (3.19) defined and smooth on a neighborhood of $y = 0$. In particular, all of these solutions are smooth at $y = 0$ and have the same Taylor series,

$$u(y) = -y - 2\,y^2 - 12\,y^3 - \cdots. \tag{3.21}$$

By continuity, exactly one of these solutions has the property that it extends over the whole interval $0 \le y \le 1$ and satisfies $\lim_{y\to 1^-} u(y) = 0$. The point $(y, z) = (1, 0)$ is a hyperbolic fixed point of the flow (3.16) and one easily shows that this solution is analytic at $y = 1$, with a convergent Taylor series of the form

$$u(y) = (y-1) + \tfrac{2}{5}(y-1)^2 - \tfrac{36}{175}(y-1)^3 + \cdots. \tag{3.22}$$

□

The rest of this section will be devoted to proving that, up to constant multiples the soliton just found is the only complete steady rotationally invariant soliton (without any assumption on the sign of the curvature).

3.3. **Solitons with $C = 0$.** A short calculation shows that the integral curves of (3.1) for which $C = 0$ can be parametrized in the form

$$x = c\,\frac{(2-s^2)}{(s+1)^2}\left(\frac{\sqrt{2}+s}{\sqrt{2}-s}\right)^{\sqrt{2}}, \quad y = \frac{s^2}{(2-s^2)}, \quad z = \frac{2(s+1)}{(2-s^2)}. \tag{3.23}$$

where s satisfies $s^2 < 2$, i.e., $|s| < \sqrt{2}$, and c is a constant (which must be positive for our purposes since $x = a(t)^2 > 0$). The two singular points of the 1-form (3.3) that lie on this curve in the yz-plane are at $s=-1$ and $s=0$. The corresponding steady soliton metric g then takes the form

$$g=\frac{c}{(s+1)^4}\left(\frac{\sqrt{2}+s}{\sqrt{2}-s}\right)^{\sqrt{2}}\left((\mathrm{d}s)^2+(s+1)^2(2-s^2)\,\mathrm{d}\sigma^2\right). \tag{3.24}$$

This metric is smooth on the two intervals $-\sqrt{2}<s<-1$ and $-1<s<\sqrt{2}$. On the interval $(-\sqrt{2},-1)$, the metric is complete towards $s=-1$, but not towards $s=-\sqrt{2}$. However, it cannot be extended to $s=-\sqrt{2}$ or below because the radial sectional curvature $-z/x$ goes to infinity as s approaches $-\sqrt{2}$. Similarly, on the interval $(-1,\sqrt{2})$, the metric is complete towards $s=-1$, but not towards $s=\sqrt{2}$. Again, it cannot be extended past $s=\sqrt{2}$ because the curvatures of this metric tend to infinity as s approaches $\sqrt{2}$. Note that neither of these solutions with $C=0$ has nonnegative sectional curvature.

3.4. **Solitons with $C\neq 0$.** To study the solutions that satisfy $C\neq 0$, the following device seems to be the most convenient: The form of the equation (3.3) is that of an Abel equation, when z is regarded as a function of y. This suggests (since I am only concerned with solutions satisfying $y\geq 0$) making the following change of variables[3]

$$y=\frac{s^2}{(2-s^2)},\qquad z=\frac{2(u+s)}{u(2-s^2)}. \tag{3.25}$$

where $u\neq 0$ and $|s|<\sqrt{2}$. In order to avoid the locus $y^2+2yz-z^2+2z-1=0$, I require that $u^2-1\neq 0$. The relation (3.3) is equivalent to

$$u(1-u^2)s^2\,\mathrm{d}s-(2-s^2)(u+s)\,\mathrm{d}u=0. \tag{3.26}$$

Note that the lines $u=0$, $u=\pm 1$ and $s=\pm\sqrt{2}$ are integral curves of the relation (3.26).

With this change of variables, one finds that

$$x=-c\,\frac{(y^2+2yz-z^2+2z-1)}{y}=c\,\frac{4(1-u^2)}{u^2(2-s^2)}, \tag{3.27}$$

where the constant c must be chosen so that $c(1-u^2)>0$.

Computation now shows that, on an integral curve of (3.26) other than $u=0$ or $u=\pm 1$, the corresponding soliton metric can be expressed in the form

$$g=c\,\frac{4(1-u^2)}{u^2(2-s^2)}\left(\frac{u^2\,(\mathrm{d}s)^2}{(u+s)^2(2-s^2)}+\mathrm{d}\sigma^2\right). \tag{3.28}$$

To analyze the integrals for which $|u|>1$, it is useful to set $u=1/v$ and express everything in terms of s and v. The relation (3.3) is then equivalent to

$$(1-v^2)s^2\,\mathrm{d}s-(2-s^2)(1+sv)\,\mathrm{d}v=0. \tag{3.29}$$

Note that the lines $s=\pm\sqrt{2}$ and $v=\pm 1$ are integral curves of the relation (3.29).

[3]Actually, a double covering, since the points (s,u) and $(-s,-u)$ represent the same yz-point. However, this covering presents an advantage, especially for understanding the solutions with $|u|>1$.

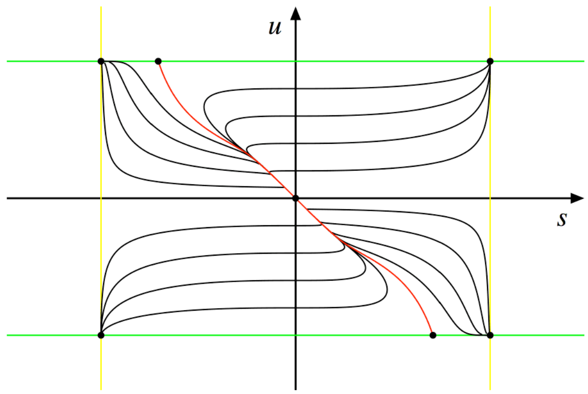

FIGURE 3. Integral curves of the relation (3.26) that lie in the rectangle $|s| < \sqrt{2}$ and $|u| < 1$. The red curves generate the complete soliton. The corners of the rectangle are sinks, the center is a source, and the points $(s, u) = (\pm 1, \mp 1)$ are hyperbolic saddles. See the text for further discussion.

Computation now shows that, on an integral curve of (3.29) that satisfies $|v| < 1$, the corresponding soliton metric can be expressed in the form

$$g = (-c)\,\frac{4(1-v^2)}{(2-s^2)}\left(\frac{(\mathrm{d}s)^2}{(1+sv)^2(2-s^2)} + \mathrm{d}\sigma^2\right). \tag{3.30}$$

The expressions for the sectional curvatures in terms of these coordinates are

$$\frac{1-y}{x} = \frac{u^2(1-s^2)}{2c(1-u^2)} = -\frac{(1-s^2)}{2c(1-v^2)} \tag{3.31}$$

and

$$\frac{-z}{x} = \frac{-u(u+s)}{2c(1-u^2)} = \frac{(1+sv)}{2c(1-v^2)}. \tag{3.32}$$

Now, these sectional curvature functions become infinite at the corners of the rectangles $|s| \le \sqrt{2}$, $|u| \le 1$ (in the su-plane) and $|s| \le \sqrt{2}$, $|v| \le 1$ (in the sv-plane). Each integral curve of (3.26) in this su-rectangle other than the red separatrices (which generate the complete soliton already found) or those on the line $u = 0$ (which correspond to no soliton) has at least one end in a corner. Similarly, each integral curve of (3.29) in the sv-rectangle has at least one end in one of the corners $(s, v) = (\pm\sqrt{2}, \pm 1)$. Thus, it suffices to show that the soliton metrics generated by integral curves ending in corners of these rectangles have an incomplete 'end' corresponding to the corner limit. Because of the invariance of the equations under the involutions $(s, u) \to (-s, -u)$ and $(s, v) \to (-s, -v)$, it suffices to examine one or two of the corners in each rectangle.

To see the incompleteness, note that the 'radial parameter' t in the metric $g = \mathrm{d}t^2 + a(t)^2\mathrm{d}\sigma^2$ takes one of the following forms (up to sign) near $|s| = \sqrt{2}$:

$$\mathrm{d}t = \frac{2|c|^{1/2}\sqrt{(1-u^2)}\,\mathrm{d}s}{(u+s)(2-s^2)} = \frac{2|c|^{1/2}\sqrt{(1-v^2)}\,\mathrm{d}s}{(1+sv)(2-s^2)}. \tag{3.33}$$

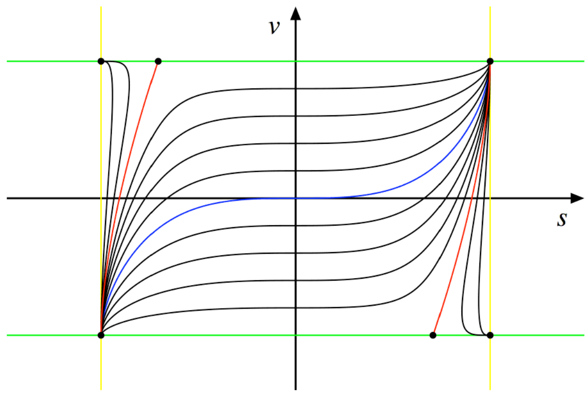

FIGURE 4. Integral curves of the relation (3.29) that lie in the rectangle $|v| < 1$ and $|s| < \sqrt{2}$. The corners of this rectangle are hyperbolic sinks. The points $(s,v) = (\pm 1, \mp 1)$ are hyperbolic saddles. See the text for further discussion.

Now, $(s,u) = \bigl(-\sqrt{2}, 1\bigr)$ is a hyperbolic singularity of the ODE. By computation, one finds that the integral curves in the rectangle $|s| < \sqrt{2}$, $|u| < 1$ that limit to this point can be expressed in the form

$$u(s) = 1 - U(s)\,\bigl(s + \sqrt{2}\bigr)^{2+\sqrt{2}} \tag{3.34}$$

for some function U continuous and positive on some s-interval $\bigl[-\sqrt{2}, -\sqrt{2}+\epsilon\bigr)$. Thus, near such an end, one has

$$\mathrm{d}t = U_1(s)\bigl(s + \sqrt{2}\bigr)^{\sqrt{2}/2}\,\mathrm{d}s. \tag{3.35}$$

where U_1 is continuous and positive on the s-interval $\bigl[-\sqrt{2}, -\sqrt{2}+\epsilon\bigr)$. The integral of this differential over this interval is finite.

Next, $(s,u) = \bigl(\sqrt{2}, 1\bigr)$ is also a hyperbolic singularity of the ODE. By computation, one finds that the integral curves in the rectangle $|s| < \sqrt{2}$, $|u| < 1$ that limit to this point can be expressed in the form

$$u(s) = 1 - U(s)\,\bigl(\sqrt{2} - s\bigr)^{2-\sqrt{2}} \tag{3.36}$$

for some function U that is continuous and positive on some s-interval $\bigl(\sqrt{2}-\epsilon, \sqrt{2}\bigr]$. Thus, near such an end, one has

$$\mathrm{d}t = U_1(s)\bigl(\sqrt{2} - s\bigr)^{-\sqrt{2}/2}\,\mathrm{d}s. \tag{3.37}$$

where U_1 is continuous and positive on the s-interval $\bigl(\sqrt{2}-\epsilon, \sqrt{2}\bigr]$. The integral of this differential over this interval is also finite since $-\sqrt{2}/2 > -1$.

In the sv-rectangle, it suffices to examine the curves limiting to $(s,v) = \bigl(\sqrt{2}, 1\bigr)$. Again, this is a hyperbolic singularity of the ODE. By computation, one finds that the integral curves in the rectangle $|s| < \sqrt{2}$, $|v| < 1$ that limit to this point can be expressed in the form

$$v(s) = 1 - V(s)\,\bigl(\sqrt{2} - s\bigr)^{2-\sqrt{2}} \tag{3.38}$$

for some function V that is continuous and positive on some s-interval $\big(\sqrt{2}-\epsilon, \sqrt{2}\big]$. Thus, near such an end, one has

$$\mathrm{d}t = V_1(s)\big(\sqrt{2}-s\big)^{-\sqrt{2}/2}\,\mathrm{d}s. \tag{3.39}$$

where V_1 is continuous and positive on the s-interval $\big(\sqrt{2}-\epsilon, \sqrt{2}\big]$. The integral of this differential over this interval is also finite since $-\sqrt{2}/2 > -1$.

Thus, all of these corners are reached in finite geodesic distance, but the sectional curvatures approach infinity as one approaches such a corner. Consequently, these solutions, although incomplete, cannot be extended any further.

Theorem 1 (Steady soliton uniqueness)**.** *Up to constant multiples, there is only one complete, steady, rotationally invariant soliton in dimension* 3 *that is not flat. It has positive sectional curvature, which reaches a maximum at the center of rotation.*

Writing the complete soliton whose maximal sectional curvature is $\frac{1}{6}$ *in the form* $g = \mathrm{d}t^2 + a(t)^2\,\mathrm{d}\sigma^2$, *where* t *is the distance from the center of rotation, one has, for large* t, *the following asymptotic orders: The aperature* $a(t)$ *has leading order term* $\sqrt{2t}$, *the radial sectional curvature has leading order term* $1/(4t^2)$, *and the orbital sectional curvature has leading order term* $1/(2t)$.

4. The expanding cases

Throughout this section, it will be assumed that $\lambda > 0$, so that the solitons to be considered will be expanding solitons.

The object is to prove the existence of a 1-parameter family of complete expanding solitons for each expansion constant $\lambda > 0$. In particular, it will be shown that (as suggested by Figure 1) each of the solutions of (2.9) that satisfy $h(0) = 1$ and $h'(0) < 0$ extends to the whole interval $0 \le r < \infty$ and corresponds to a complete expanding soliton of positive sectional curvature that decays to 0 as distance from the center of rotation (i.e., $r = 0$) goes to infinity. As will be seen, these solutions are asymptotic to cones, with the curvatures falling off as the square of the reciprocal of the distance to the center of symmetry.

It will first be necessary to develop a few properties of solutions to (2.9).

Lemma 1 (No oscillation)**.** *Let* h *be a positive solution of* (2.9) *defined on an* r*-interval* $(L, M) \subset (0, \infty)$. *Either* $h \equiv 1$ *or else* h *has at most one critical point on* (L, M), *which will be nondegenerate if it exists. Moreover, if* $h'(r_0) \ge 0$ *for some* $r_0 \in (L, M)$ *and* $h(r_0) > 1$, *then* h' *is positive on* (r_0, M) *while if* $h'(r_0) \le 0$ *for some* $r_0 \in (L, M)$ *and* $h(r_0) < 1$, *then* h' *is negative on* (r_0, M).

Proof. Suppose that $h'(r_0) = 0$ for some $r_0 \in (L, M)$. If $h(r_0) = 1$, then $h(r) = 1$ for all $r \in (L, M)$ by ODE uniqueness. Setting this case aside, one sees by (2.9) that

$$h''(r_0) = \frac{h(r_0) - 1}{2{r_0}^2} \ne 0, \tag{4.1}$$

so that each critical point of h is nondegenerate (and hence isolated). If $h(r_0) > 1$, then $h''(r_0) > 0$, which implies that, if there be another critical point of h and $r_1 \in (L, M)$ be one that is 'adjacent' to r_0 (i.e., h' has no zeroes between r_0 and r_1), then $h(r_1) > h(r_0) > 1$, implying that $h''(r_1) > 0$ so that h also has a strict local minimum at r_1. Evidently, this is impossible. Similarly, if $h(r_0) < 1$, then $h''(r_0) < 0$, which implies that, if there be another critical point of h and $r_1 \in (L, M)$ be one that is 'adjacent' to r_0 (i.e., h' has no zeroes between r_0 and r_1), then $h(r_1) < h(r_0) < 1$, implying that $h''(r_1) < 0$ so that h also has a strict local maximum at r_1. Again, this is impossible. Thus, h can have at most one critical point in (L, M).

Suppose that $h'(r_0) \leq 0$ and $h(r_0) < 1$. If $h'(r_0) = 0$ then h has a nondegenerate local maximum at r_0. Moreover, h' cannot have any other zeros in (L, M) and hence must be strictly negative on (r_0, M). If $h'(r_0) < 0$ and yet $h'(r_1) = 0$ for some $r_1 \in (r_0, M)$ then h' would then have to be positive just before r_1 and hence would have to have another zero between r_0 and r_1. Thus, $h' < 0$ on (r_0, M).

The case when $h'(r_0) \geq 0$ and $h(r_0) > 1$ is entirely analogous. $\square$

4.1. **A projective interpretation.** At this point, it is useful to give a geometric interpretation of (2.9), namely, as an equation for the geodesics of a projective structure. Note that the right hand side of (2.9) is a polynomial of degree two in $h'(r)$. Since (2.9), for $r > 0$ and $h(r) > 0$, can be written in the form

$$h''(r) = \frac{(h(r) - 1)}{2r^2} - \frac{(1 + \lambda\, r)}{2rh(r)}\, h'(r) + \frac{\big(h'(r)\big)^2}{2h(r)} \tag{4.2}$$

it can thought of as defining the geodesics of a projective structure on the (open) first quadrant of the rh-plane. The geodesics of this projective structure consist of the curves of the form $\big(r, h(r)\big)$, where h is a positive solution of (2.9) defined on an r-interval inside $(0, \infty)$ together with the vertical lines $r = r_0$ (since there is no $h'(r)^3$ term in the right hand side).

One immediate consequence of this interpretation is that, because the vertical lines $r = r_0$ are geodesics of this projective structure, any other geodesic is transverse to the vertical lines along its entire length. In other words, each (non-vertical) maximally extended geodesic can be parametrized in the form $\big(r, h(r)\big)$ where h is a positive solution of (2.9) defined on an interval (L, M) where $0 \leq L < M \leq \infty$ and where, if $L > 0$ then $\log h(r)$ becomes unbounded as $r \to L^+$ and, if $M < \infty$, then $\log h(r)$ becomes unbounded as $r \to M^-$.

Proposition 4 (Long time existence)**.** *Let $h : (L, M) \to \mathbb{R}^+$ be a maximally extended solution to* (2.9) *for $(L, M) \subset \mathbb{R}^+$. Then $M = \infty$.*

Proof. Suppose that $M < \infty$. Since $\log h(r)$ becomes unbounded as $r \to M^-$, and since h is eventually monotone, either $\lim_{r\to M^-} h(r) = 0$ or else $\lim_{r\to M^-} h(r) = \infty$.

If $\lim_{r\to M^-} h(r) = 0$, then there must exist an $r_0 \in (L, M)$ such that $h(r_0) < 1$ and $h'(r_0) < 0$. By Lemma 1, it follows that h' is strictly negative on (r_0, M). The equation (4.2) then implies that, for $r_0 \leq r < M$, one has

$$h''(r) > -\frac{1}{2r^2} - \frac{\lambda}{2}\,\frac{h'(r)}{h(r)}. \tag{4.3}$$

Integrating this inequality from r_0 to $r < M$ yields

$$h'(r) - h'(r_0) > \frac{1}{2}\left(\frac{1}{r} - \frac{1}{r_0}\right) + \frac{\lambda}{2}\,\log\left(\frac{h(r_0)}{h(r)}\right). \tag{4.4}$$

The left hand side of this inequality is bounded above by $-h'(r_0)$ (since $h'(r) < 0$ for $r_0 \leq r < M$) while the right hand side must approach $+\infty$ as $r \to M^-$ increases (since $\lim_{r\to M^-} h(r) = 0$ and $\lambda > 0$). This is a contradiction.

On the other hand, if $\lim_{r\to M^-} h(r) = \infty$, then there must exist an $r_0 \in (L, M)$ such that $h(r_0) > 1$ and $h'(r_0) > 0$. By Lemma 1, it follows that h' is strictly positive on (r_0, M). The equation (4.2) then implies that, for $r_0 \le r < M$, one has

$$h''(r) - \frac{\big(h'(r)\big)^2}{2h(r)} < \frac{h(r)}{2\,{r_0}^2}. \tag{4.5}$$

Multiplying both sides of this inequality by the positive quantity $2h'/h$ and integrating from r_0 to $r < M$ yields

$$\frac{\big(h'(r)\big)^2}{h(r)} - \frac{\big(h'(r_0)\big)^2}{h(r_0)} < \frac{\big(h(r) - h(r_0)\big)}{{r_0}^2}. \tag{4.6}$$

Simple rearrangement then yields

$$\sqrt{\frac{{r_0}^2\,h(r_0)}{h(r)\big((r_0h'(r_0))^2 + h(r_0)\big(h(r) - h(r_0)\big)\big)}}\ h'(r) < 1. \tag{4.7}$$

By hypothesis $h(r)$ goes to infinity as r approaches M, so integrating this inequality from r_0 to M yields

$$\int_{h(r_0)}^{\infty} \sqrt{\frac{{r_0}^2\,h(r_0)}{h\big((r_0h'(r_0))^2 + h(r_0)\big(h - h(r_0)\big)\big)}}\ \mathrm{d}h < M - r_0. \tag{4.8}$$

However, the integral on the left hand side of this inequality is infinite. This is a contradiction.

Thus, it has been shown that $M = \infty$. □

Remark 9 (Positive lower limits). In contrast with this 'big r existence' result for the upper limit M, it can happen that $L > 0$. For example, the positive solutions h of the form (2.37) have $L = a_0^2 > 0$.

Corollary 2. *Each solution h of* (2.9) *that satisfies $h(0) = 1$ and $h'(0) < 0$ exists for all $r > 0$ and has a positive lower bound.*

Proof. That the solution exists for all $r \ge 0$ follows immediately from Proposition 4. By Lemma 1, h must be strictly monotone decreasing, so that, as in the proof of Proposition 4, one has the inequality

$$h'(r) - h'(r_0) > \frac{1}{2}\left(\frac{1}{r} - \frac{1}{r_0}\right) + \frac{\lambda}{2}\,\log\left(\frac{h(r_0)}{h(r)}\right). \tag{4.9}$$

for all $0 < r_0 < r$. However, the left hand side of this inequality is bounded above by $-h'(r_0)$ while the right hand side cannot be bounded above if h decreases to 0 as $r \to \infty$. Thus, h must have a positive lower bound as $r \to \infty$. □

Proposition 5 (Complete, positive curvature expanders)**.** *Each of the solutions $h : [0,\infty) \to (0,1]$ to* (2.9) *with $h'(0) < 0$ defines a rotationally invariant, complete soliton with expansion constant $\lambda > 0$ and positive sectional curvature via the formula*

$$g = \frac{\mathrm{d}a^2}{h(a^2)} + a^2\,\mathrm{d}\sigma. \tag{4.10}$$

The sectional curvatures reach a maximum value of $-h'(0) > 0$ at $a = 0$ and the maximum sectional curvature decays at a rate proportional to the reciprocal of the square of the distance from the center.

Remark 10 (Asymptotics). The distance t from the center of rotation is given by the integral

$$t = \int_0^a \frac{\mathrm{d}\tau}{\sqrt{h(\tau^2)}}\,. \tag{4.11}$$

For small r, one has $h(r) = 1 + h'(0)\,r + O(r^2)$ so, for small a,

$$t = a - \tfrac{1}{6}h'(0)\,a^3 + O(a^4). \tag{4.12}$$

On the other hand, for large r, when $\lim_{r\to\infty} h(r) = 1-\rho > 0$ for some $\rho > 0$, it is not difficult to prove (see Remark 11) that one one has the (nonconvergent) asymptotic expansion

$$h(r) \sim (1{-}\rho) + \frac{\rho(1{-}\rho)}{\lambda r} + \frac{\rho(1{-}\rho)^2}{(\lambda r)^2} + \cdots. \tag{4.13}$$

(Note that because h is asymptotic to a positive constant, the metric g is asymptotic to a cone.) This leads to the (also nonconvergent) asymptotic expansion for large a

$$t \sim \frac{a}{\sqrt{1{-}\rho}} + \nu + \frac{\rho}{2\lambda\sqrt{1{-}\rho}\,a} + \cdots \tag{4.14}$$

where $\nu \le 0$ is a constant.

In particular, note that the radial sectional curvature for large a is

$$-h'(a^2) \sim \frac{\rho(1{-}\rho)}{\lambda a^4} + \frac{2\rho(1{-}\rho)^2}{\lambda^2 a^6} + \cdots \simeq \frac{\rho}{\lambda(1{-}\rho)\,t^4}, \tag{4.15}$$

while the orbital sectional curvature is

$$\frac{1-h(a^2)}{a^2} \sim \frac{\rho}{a^2} - \frac{\rho(1{-}\rho)}{\lambda a^4} - \frac{\rho(1{-}\rho)^2}{\lambda^2 a^6} + \cdots \simeq \frac{\rho}{(1{-}\rho)\,t^2}. \tag{4.16}$$

Remark 11 (Asymptotics of h at infinity). If h is a solution of (2.9) that is defined on an r-interval of the form (L,∞) and has a finite limit as r tends to infinity, then h has an asymptotic expansion of the form

$$h(r) \sim \sum_{j=0}^{\infty} \frac{c_j}{(\lambda r)^j}. \tag{4.17}$$

This formal series satisfies the equation if and only if the coefficients c_j satisfy the recursion

$$(j{+}1)c_{j+1} + (j{-}1)c_j - \sum_{p+q=j} (p^2{-}pq{+}q^2{+}p{+}q{-}1)c_p c_q = 0, \qquad j \ge 0. \tag{4.18}$$

One can show that, except when $c_0 = 0$ or $c_0 = 1$, this formal series has zero radius of convergence. However, the asymptotics do give useful information.

The positively curved examples produced above limit to the flat metric as the sectional curvature at the center decreases to zero. Interestingly, enough, one can continue this family through the flat solution to negatively curved complete examples, whose curvatures still have the same quadratic decay rate.

Lemma 2 (Increasing bounded solutions). *Let $h : [0,\infty) \to \mathbb{R}$ be a solution of (2.9) that satisfies $h(0) = 1$ and $0 < h'(0) < \frac{3}{7}\lambda$. Then h is strictly increasing and $\lim_{r\to\infty} h(r) < \infty$.*

Corollary 3 (Complete, asymptotically conical, negatively curved expanders). *There exist complete, negatively curved, rotationally invariant expanding solitons whose curvature decays at least quadratically with distance from the center of rotation.*

Proof. Just use the bounded solutions h described in Lemma 2 in the formula (4.10). The boundedness of h implies completeness of the metric (and the asymptotic expansion for h near $r = \infty$ shows that the metric g is asymptotically conical). Since $h(r) > 1$ for $r > 0$ and $h'(r) > 0$ for all $r \geq 0$, the sectional curvatures of the metric are negative. The asymptotic formulae still hold as before, so the relations between the distance t from the center of rotation, the aperature a, and the sectional curvatures hold as before (except that, now, $\rho < 0$). □

Remark 12 (A possibly improvable bound). Note that, because $h(r) = 1 + \frac{1}{2}\lambda r$ is a solution (which, by Remark 6, corresponds to the space form of constant sectional curvature $-\frac{1}{2}\lambda < 0$), some upper bound on $h'(0)$ must be assumed in order for the boundedness conclusion in Lemma 2 to hold. Perhaps the best upper bound in Lemma 2 is really $\frac{1}{2}\lambda$ rather than $\frac{3}{7}\lambda$, but the proof to be given below cannot be strengthened to get this.

Proof. (of Lemma 2.) The idea is based on the notion of a *subsolution* of (2.9), which is defined to be a twice differentiable function g defined on an r-interval $I \subset (0, \infty)$ that satisfies

$$2r^2 g(r)g''(r) > g(r)\big(g(r) - 1\big) + rg'(r)\big(rg'(r) - \lambda r - 1\big) \tag{4.19}$$

for all $r \in I$.

For example, for any numbers $a, b > 0$ consider the function

$$g_{a,b}(r) = \frac{1 + b\lambda r}{1 + a\lambda r}. \tag{4.20}$$

This function is defined and positive on the r-interval $[0, \infty)$ and satisfies

$$\begin{aligned} 2r^2 g_{a,b}(r)g''_{a,b}(r) - g_{a,b}(r)\big(g_{a,b}(r){-}1\big) + rg'_{a,b}(r)\big(rg'_{a,b}(r){-}\lambda r{-}1\big) \\ = \frac{(b-a)(\lambda r)^2\big((1{-}b)(a\lambda r)^2 + 2(1{-}3b)(a\lambda r) + 1{-}3a{-}2b\big)}{\big(1 + a\lambda r\big)^4}. \end{aligned}$$

The righthand side of this equation is positive for all $r > 0$ when $b > a > 0$ and $1{-}3a{-}2b > 0$ and either $b \leq \frac{1}{3}$ or $\frac{1}{3} \leq b < \frac{3}{7}$ and $0 < a < b(3{-}7b)/\big(3(1{-}b)\big)$. Thus, for (a, b) satisfying these restrictions, $g_{a,b}$ is a subsolution of (2.9) on the r-interval $(0, \infty)$. For use below, note that, when (a, b) satisfies the above restrictions, the function $g_{a,b}$ is strictly increasing for positive r and that

$$\lim_{r\to\infty} g_{a,b}(r) = \frac{b}{a} \qquad \text{and} \qquad g_{a,b}(r) = 1 + (b{-}a)\,\lambda r - a(b{-}a)\,(\lambda r)^2 + O(r^3). \tag{4.21}$$

Now, $(a, b) = \big(\frac{1}{7}c, \frac{1}{7}(3{-}2c)\big)$ with $0 < c < 1$ satisfies these restrictions. Consider the mapping $\Phi : [0, \infty) \times [0, 1] \to \mathbb{R}^2$ defined by

$$\Phi(r, c) = \left(r,\ \frac{1 + \frac{1}{7}(3{-}2c)\,\lambda r}{1 + \frac{1}{7}c\,\lambda r}\right). \tag{4.22}$$

Note that $\Phi(r,1) = (r,1)$ and $\Phi(r,0) = \bigl(r, 1+\frac{3}{7}\lambda r\bigr)$ while, for $0 < c < 1$, the curve $r \mapsto \Phi(r,c)$ is the graph of an increasing subsolution $g_{a,b}$ whose Taylor expansion at $r = 0$ is of the form

$$g_{a,b}(r) = 1 + \tfrac{3}{7}(1{-}c)\,\lambda r - \tfrac{3}{49}c(1-c)\,(\lambda r)^2 + O(r^3). \tag{4.23}$$

and that satisfies

$$\lim_{r\to\infty} g_{a,b}(r) = \frac{3{-}2c}{c}\,. \tag{4.24}$$

Moreover, the mapping $\Phi : (0,\infty)\times(0,1) \to \mathbb{R}^2$ is a diffeomorphism onto its image, namely the wedge

$$W = \bigl\{(r,h) \mid 1 < h < 1+\tfrac{3}{7}\lambda r\,\bigr\}. \tag{4.25}$$

The curves $r \mapsto \Phi(r,c)$ foliate W by graphs of subsolutions.

Now, consider a solution h of (2.9) that satisfies $h(0) = 1$ and $0 < h'(0) < \frac{3}{7}\lambda$. It has already been shown that h is defined and strictly increasing (since h' is positive) for all $r > 0$. In particular, $h(r) > 1$ for all $r > 0$.

Because of the inequality $h'(0) < \frac{3}{7}\lambda$, there is some largest $R \le \infty$ such that $h(r) < 1+\frac{3}{7}\lambda r$ for all $r < R$. If $R < \infty$, note that this implies $h(R) = 1+\frac{3}{7}\lambda R$. (It will eventually be seen that $R = \infty$, though.) Thus, for $0 < r < R$, there exists a $c(r) \in (0,1)$ such that $\bigl(r, h(r)\bigr) = \Phi\bigl(r, c(r)\bigr)$. Defining $c(0) \in (0,1)$ so that $h'(0) = \frac{3}{7}\bigl(1{-}c(0)\bigr)\lambda$, one sees that c is continuous at 0 and, because $\Phi\bigl(r, c(0)\bigr) = \bigl(r, g(r)\bigr)$ where g is a subsolution of (2.9), it follows easily that $h(r) < g(r)$ for all sufficiently small $r > 0$.

I claim that, in fact, c is a strictly increasing function of r. To see this, note that $c'(0) > 0$ (because $g''(0) > h''(0)$) and consider what happens at a point $r_0 > 0$ such that $c'(r_0) = 0$. For such an r_0, the curve $r \mapsto \Phi\bigl(r, c(r_0)\bigr) = \bigl(r, \bar g(r)\bigr)$ is tangent to the curve $r \mapsto \bigl(r, h(r)\bigr)$ at $\bigl(r_0, h(r_0)\bigr)$ and, because $\bar g$ is a subsolution satisfying $\bar g(r_0) = h(r_0)$ and $\bar g'(r_0) = h'(r_0)$, must therefore satisfy $\bar g''(r_0) > h''(r_0)$. In particular, the graph of $\bar g$ near r_0 must lie strictly above the graph of h near r_0. Thus $c''(r_0) > 0$, i.e., the only critical points of c are strict local minima. Since $c'(0) > 0$, this implies that c cannot have any critical points in the range $(0,R)$. Thus c is always strictly increasing, but is bounded above by 1, In fact, though, it cannot reach the value 1 because h is strictly increasing and $\Phi(r,1) = (r,1)$. Thus, $R < \infty$ leads to a contradiction since then c could be extended continuously to $[0,R]$ with $c(R) < 1$, implying that $\bigl(R, h(R)\bigr) = \Phi\bigl(R, c(R)\bigr)$, so that $h(R) < 1 + \frac{3}{7}\lambda R$, violating the definition of R.

In particular, it now follows that $\lim_{r\to\infty} c(r) = c_\infty$ for some $c_\infty \in \bigl(c(0), 1\bigr)$, so that

$$\lim_{r\to\infty} h(r) = \frac{3-2c_\infty}{c_\infty} < \frac{3-2c(0)}{c(0)} = \frac{\frac{3}{7}\lambda + 2h'(0)}{\frac{3}{7}\lambda - h'(0)}. \tag{4.26}$$

Remark 13 (Apologia). The proof of Lemma 2 certainly appears to be rather *ad hoc* and the reader will undoubtedly wonder whether it can be improved by a more judicious choice of subsolution field than that afforded by (4.22). I certainly have tried to come up with one, but so far, this is the best I have found. There is, of course, a subsolution field that covers the whole wedge $1 \le h \le 1 + \frac{1}{2}\lambda r$, namely

$$\Psi(r,c) = \bigl(r, 1+\tfrac{1}{2}c\,\lambda r\bigr), \qquad r \ge 0,\ 0 \le c \le 1. \tag{4.27}$$

However, this subsolution field is only good for proving that, for any solution h of (2.9) that satisfies $h(0) = 1$ and $0 < h'(0) < \frac{1}{2}\lambda$, the quantity $\big(h(r)-1\big)/r$ is strictly decreasing in $r > 0$. In particular, such a solution, while increasing, must have sublinear growth.

□

5. The shrinking cases

This section has yet to be written up.

The first step is to study the maximal extent of a positive solution to (2.9) when $r > 0$ and $\lambda < 0$. Note that Lemma 1 continues to hold in this case since the sign of λ was not used in that discussion.

When $\lambda < 0$, one cannot hope that all positive solutions of (2.9) extend to a neighborhood of $r = \infty$; just look at the solutions (2.37) where ${a_0}^2 > -1/\lambda$, which are positive on an interval of the form $\big(L, {a_0}^2\big)$. However, this is, in some sense, the worst case:

Lemma 3. *Any maximally extended solution $h : (L, M) \to \mathbb{R}^+$ to (2.9) with $\lambda < 0$ and $M > 0$ has $M \geq -1/\lambda$.*

Proof. Suppose that $h : (L, M) \to \mathbb{R}^+$ were such a maximally extended solution and $0 < M \leq -1/\lambda$. I will show that, in this case $M = -1/\lambda$.

Just as before in the discussion of projective geodesics, $h(r)$ either increases to $+\infty$ or decreases to 0 as r approaches M (from below) and there is an $r_0 \in (0, M)$ such that h' is nonvanishing on (r_0, M).

As in the proof of Proposition 4, if h increases to $+\infty$ as $r \to M^-$ one reaches a contradiction using the inequality (4.5) (which remains valid because the hypothesis that $M \leq -1/\lambda$ yields that $1 + \lambda r > 0$ on the interval (r_0, M)).

Thus, it suffices to treat the case in which h decreases to 0 as r approaches M. By choosing $r_0 > M$ sufficiently close to M, one can assume both that h' is negative on (r_0, M) and $h(r) < 1$ for $r_0 \leq r < M$. Now, one has the inequality

$$h''(r) > -\frac{1}{2r^2} - \frac{1+\lambda r}{2r}\,\frac{h'(r)}{h(r)}. \tag{5.1}$$

Integrating this inequality from r_0 to $r \in (r_0, M)$ (and integrating by parts) yields

$$h'(r)-h'(r_0) > \frac{1}{2}\left(\frac{1}{r}-\frac{1}{r_0}\right)-\frac{1+\lambda r}{2r}\,\ln h(r)+\frac{1+\lambda r_0}{2r_0}\,\ln h(r_0)-\int_{r_0}^{r}\frac{\ln h(\rho)}{2\rho^2}\,\mathrm{d}\rho. \tag{5.2}$$

The left hand side of this inequality is bounded above as r approaches M, so the right hand side must be as well. Because the function $1 + \lambda r$ is positive on the interval (r_0, M) and because the function h is less than 1 on this interval, each of the right hand side terms

$$-\frac{1+\lambda r}{2r}\,\ln h(r) \qquad \text{and} \qquad -\int_{r_0}^{r}\frac{\ln h(\rho)}{2\rho^2}\,\mathrm{d}\rho \tag{5.3}$$

must be positive and bounded above as r approaches M. If $M < -1/\lambda$ held, then the first of these two terms would approach $+\infty$ as r approaches M, which is a contradiction. Thus, $0 < M \leq -1/\lambda$ implies $M = -1/\lambda$. Consequently, $M > 0$ implies $M \geq -1/\lambda$, as desired. □

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