ON CANONICAL CONNECTIONS FOR G-STRUCTURES

ROBERT L. BRYANT

Abstract. These are my notes on the question of the existence and uniqueness of canonical connections for G-structures. It mainly consists of two cautionary examples.

Contents

1. Introduction

For many of the G-structures that arise in differential geometry, the main method of computing local invariants of such structures is to introduce a connection in an intrinsic way and look at the covariant derivatives of the various tensors and generalized tensors that appear in the problem being studied.

Of course, every G-structure on a manifold (always assumed to be paracompact and smooth) admits *some* compatible connection, but, usually, one wants to choose such a connection that is well-suited to the study of the given G-structure. In fact, for a given $G \subset \mathrm{GL}(n, \mathbb{R})$, one often wants to be able to define a 'canonical' choice of connection for each G-structure B on an n-manifold, one that has the virtue of being preserved under equivalence of G-structures.

Perhaps a more descriptive adjective than 'canonical' would be 'functorial': Formally, one wants to define a functor from the category whose objects are smooth n-manifolds endowed with G-structures (M, B) and whose morphisms are diffeomorphisms of the underlying manifolds that identify the corresponding G-structures to the category whose objects are smooth n-manifolds endowed with affine connections (M, ∇) and whose morphisms are diffeomorphisms of the underlying manifolds

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that identify the corresponding affine connections. However, the common usage in differential geometry is to describe such a functor as a 'canonical choice' of connection.

For example, in Riemannian geometry, i.e., the geometry of $\mathrm{O}(n)$-structures on n-manifolds, one usually introduces and uses the Levi-Civita connection, which is characterized by simple conditions derived from the given $\mathrm{O}(n)$-structure, namely that the covariant derivative of the underlying metric should vanish and that the torsion should vanish. (More generally, this construction can be used to define a canonical choice of connection for any $G \subset \mathrm{GL}(n,\mathbb{R})$ whose closure is compact.)

Nevertheless, it is easy to see that not every $G \subset \mathrm{GL}(n,\mathbb{R})$ allows a canonical choice of connection for G-structures. For example, if $G \subset \mathrm{GL}(n,\mathbb{R})$ is such that there exists a G-structure on some n-manifold whose symmetry group contains nontrivial elements that fix a point up to second order, such a symmetry group cannot leave invariant any affine connection, and hence there cannot be any canonical choice of affine connection for G-structures on n-manifolds. For example, this happens in the cases of (almost) symplectic geometry and conformal geometry.

At first glance, it might seem reasonable to suppose that this sort of obstruction is the only obstruction, namely that if $G \subset \mathrm{GL}(n,\mathbb{R})$ is such that no G-structure on an n-manifold admits a (local) symmetry (pseudo-)group that contains nontrivial elements that fix a point to second order,[1] then there should be a way to define a 'canonical' connection for each such G-structure.

However, it turns out that this is not the case. Moreover, even in the case that there does exist a 'canonical' connection, it may well not be unique in the sense that there could be more than one 'recipe' that yields a 'canonical' connection. The point of these notes is to give some examples of these phenomena in the simplest cases, when $n = 3$.

2. Lack of Uniqueness

First, I will consider a case in which there always exists a 'canonical' connection, but in which there may be several possible choices.

Let λ_1, λ_2, and λ_3 be nonzero real numbers, fixed throughout this example. Consider the connected subgroup $G \subset \mathrm{GL}(3,\mathbb{R})$ of dimension 2 that consists of the diagonal matrices with positive entries of the form

$$\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \qquad \text{with} \qquad (t_1)^{\lambda_1}(t_2)^{\lambda_2}(t_3)^{\lambda_3} = 1.$$

Let M be a 3-manifold and let $\pi : B \to M$ be a G-structure on M. In other words, $\pi : B \to M$ is a principal right G-bundle where an element $u \in B$ that satisfies $\pi(u) = x$ is a linear isomorphism $u : T_xM \to \mathbb{R}^3$, and the right action of $g \in G$ satisfies $u \cdot g = g^{-1} \circ u$.

Define the tautological 1-forms ω^i on B by the usual rule

$$\begin{pmatrix} \omega^1(v) \\ \omega^2(v) \\ \omega^3(v) \end{pmatrix} = u\big(\pi'(v)\big) \qquad \text{for all } v \in T_uB.$$

[1] Equivalently, the first prolongation of $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{R})$ vanishes, where the first prolongation $\mathfrak{g}^{(1)}$ is defined as the intersection of $\mathfrak{g} \otimes (\mathbb{R}^n)^* \subset \big(\mathbb{R}^n \otimes (\mathbb{R}^n)^*\big) \otimes (\mathbb{R}^n)^*$ with $\mathbb{R}^n \otimes S^2\big((\mathbb{R}^n)^*\big)$.

A connection on B will then consist of a triple of 1-forms θ^i on B that are invariant under the right action by G (since G is abelian) and that satisfy

$$\lambda_1\,\theta^1 + \lambda_2\,\theta^2 + \lambda_3\,\theta^3 = 0$$

and

$$\begin{aligned}
\mathrm{d}\omega^1 &= -\theta^1\wedge\omega^1 + A^1_{23}\,\omega^2\wedge\omega^3 + A^1_{31}\,\omega^3\wedge\omega^1 + A^1_{12}\,\omega^1\wedge\omega^2\\
\mathrm{d}\omega^2 &= -\theta^2\wedge\omega^2 + A^2_{23}\,\omega^2\wedge\omega^3 + A^2_{31}\,\omega^3\wedge\omega^1 + A^2_{12}\,\omega^1\wedge\omega^2\\
\mathrm{d}\omega^3 &= -\theta^3\wedge\omega^3 + A^3_{23}\,\omega^2\wedge\omega^3 + A^3_{31}\,\omega^3\wedge\omega^1 + A^3_{12}\,\omega^1\wedge\omega^2
\end{aligned}$$

for some functions $A^i_{jk} = -A^i_{kj}$ on B.

By the usual general theorem, we know that a connection exists, but we would like to have a way to choose a unique one.

2.1. **A canonical coframing.** Now, using the fact that the λ^i are nonzero, one can show that there exist unique 1-forms τ^1, τ^2, and τ^3 on B such that

$$\lambda_1\,\tau^1 + \lambda_2\,\tau^2 + \lambda_3\,\tau^3 = 0$$

and

$$\begin{aligned}
\mathrm{d}\omega^1 &= -\tau^1\wedge\omega^1 + 2\lambda_2\lambda_3\,A^1\,\omega^2\wedge\omega^3\\
\mathrm{d}\omega^2 &= -\tau^2\wedge\omega^2 + 2\lambda_3\lambda_1\,A^2\,\omega^3\wedge\omega^1\\
\mathrm{d}\omega^3 &= -\tau^3\wedge\omega^3 + 2\lambda_1\lambda_2\,A^3\,\omega^1\wedge\omega^2
\end{aligned}$$

for some functions A^1, A^2, and A^3 on B. These forms will satisfy $\tau^i \equiv \theta^i$ mod $\omega^1, \omega^2, \omega^3$, but it has yet to be verified that the τ^i themselves define a connection on B, since we do not yet know that they are invariant under the right action of G.

However, by computing the exterior derivatives of the above equations and analyzing the results, one finds that

$$\mathrm{d}A^i = A^i(\tau^1 + \tau^2 + \tau^3 - 2\tau^i) + A^i_j\,\omega^j$$

and

$$\begin{aligned}
\mathrm{d}\tau^1 &= \lambda_2\lambda_3\left(2A^1_1\,\omega^2\wedge\omega^3 - (A^2_2 + K^2)\,\omega^3\wedge\omega^1 - (A^3_3 - K^3)\,\omega^1\wedge\omega^2\right)\\
\mathrm{d}\tau^2 &= \lambda_3\lambda_1\left(2A^2_2\,\omega^3\wedge\omega^1 - (A^3_3 + K^3)\,\omega^1\wedge\omega^2 - (A^1_1 - K^1)\,\omega^2\wedge\omega^3\right)\\
\mathrm{d}\tau^3 &= \lambda_1\lambda_2\left(2A^3_3\,\omega^1\wedge\omega^2 - (A^1_1 + K^1)\,\omega^2\wedge\omega^3 - (A^2_2 - K^2)\,\omega^3\wedge\omega^1\right)
\end{aligned}$$

for some functions A^i_j and K^i on B. It follows that the forms τ^i *are* actually invariant under the right action of G and hence define an actual connection on B.

2.2. **The possibility of other canonical connections.** Thus, a 'canonical' connection exists in this case. However, there is also the possibility (depending on the ratios of the λ_i) that there might be other choices of 'canonical' connection for these G-structures. If $(\bar\tau^1, \bar\tau^2, \bar\tau^3)$ is to be such a choice, then $\bar\tau^i = \tau^i + \rho^i$ where the ρ^i would have to be some 'canonical' 1-forms that would be well-defined on the base manifold M and hence be of the form

$$\rho^i = r^i_j\,\omega^j$$

for some functions r^i_j on B. These would have to be subject to the linear relation $\lambda_1\rho^1 + \lambda_2\rho^2 + \lambda_3\rho^3 = 0$.

In order to be 'canonical', the r^i_j would need to be some specific 'universal' functions of the A^i and K^i and, in order for the ρ^i to be π-pullbacks of forms defined on M the functions r^i_j would need to satisfy

$$\mathrm{d}r^i_j \equiv r^i_j \tau^j \mod \omega^1, \omega^2, \omega^3.$$

For example, suppose $[\lambda_1, \lambda_2, \lambda_3] = [-2, 1, 1]$. Then

$$\mathrm{d}(A^1\omega^1) \equiv (\tau^2 + \tau^3 - 2\tau^1) \wedge (A^1\omega^1) \equiv 0 \mod \omega^2 \wedge \omega^3, \omega^3 \wedge \omega^1, \omega^1 \wedge \omega^2,$$

so that $A^1\omega^1$ is the π-pullback of a well-defined 1-form on M, and one could take, for example, $(\rho^1, \rho^2, \rho^3) = (0, \lambda_3 A^1\omega^1, -\lambda_2 A^1\omega^1)$, which would yield a new 'canonical' choice $\bar\tau^i$ of connection forms on B.

Similarly, if $[\lambda_1, \lambda_2, \lambda_3] = [1, -1, 2]$, then one can check that $A^1(A^2)^2\omega^3$ is the π-pullback of a 1-form on M and hence can be used to modify the connection forms τ^i. Likewise, if $[\lambda_1, \lambda_2, \lambda_3] = [-1, 1, 1]$, then $K^1\omega^1$ is the π-pullback of a 1-form on M and is a second order invariant that can be used to modify the 'fundamental', 'canonical' connection forms τ^i. (In particular, it can happen that a 'canonical' connection might depend on higher order information about the G-structure than first order information.)

On the other hand, if the ratios of the λ_i are not rational, then no construction of this kind will work, and the τ^i constructed above give the only possibility for a canonical connection.

3. Lack of Existence

Now, I will consider a case in which there does not exist a 'canonical' connection for the general G-structure.

Consider the connected abelian subgroup $G \subset \mathrm{GL}(3, \mathbb{R})$ of dimension 2 that consists of the matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ ab & a & 0 \\ \frac{1}{2}ab^2 & ab & a \end{pmatrix} \qquad \text{with } a > 0 \text{ and } b \text{ arbitrary.}$$

Let M be a 3-manifold and let $\pi : B \to M$ be a G-structure on M. In other words, $\pi : B \to M$ is a principal right G-bundle where an element $u \in B$ that satisfies $\pi(u) = x$ is a linear isomorphism $u : T_xM \to \mathbb{R}^3$ and the right action of $g \in G$ satisfies $u \cdot g = g^{-1} \circ u$.

Define the tautological 1-forms ω^i on B by the usual rule

$$\begin{pmatrix} \omega^1(v) \\ \omega^2(v) \\ \omega^3(v) \end{pmatrix} = u\big(\pi'(v)\big) \qquad \text{for all } v \in T_uB.$$

A connection on B will then consist of a pair of 1-forms α and β on B that are invariant under the right action by G (since G is abelian) and that satisfy

$$\begin{aligned}
\mathrm{d}\omega^1 &= -\alpha\wedge\omega^1 && + A^1_{23}\,\omega^2\wedge\omega^3 + A^1_{31}\,\omega^3\wedge\omega^1 + A^1_{12}\,\omega^1\wedge\omega^2\\
\mathrm{d}\omega^2 &= -\beta\wedge\omega^1 - \alpha\wedge\omega^2 && + A^2_{23}\,\omega^2\wedge\omega^3 + A^2_{31}\,\omega^3\wedge\omega^1 + A^2_{12}\,\omega^1\wedge\omega^2\\
\mathrm{d}\omega^3 &= \quad - \beta\wedge\omega^2 - \alpha\wedge\omega^3 && + A^3_{23}\,\omega^2\wedge\omega^3 + A^3_{31}\,\omega^3\wedge\omega^1 + A^3_{12}\,\omega^1\wedge\omega^2
\end{aligned}$$

for some functions $A^i_{jk} = -A^i_{kj}$ on B.

By the usual general theorem, we know that a connection exists, but we would like to have a way to choose a unique one.

3.1. **A canonical coframing.** Now, by elementary linear algebra, there exist unique 1-forms λ and ρ on B such that

$$\begin{aligned}
\mathrm{d}\omega^1 &= -\lambda\wedge\omega^1 && + A^1\,\omega^2\wedge\omega^3 + A^2\,\omega^3\wedge\omega^1 + A^3\,\omega^1\wedge\omega^2\\
\mathrm{d}\omega^2 &= -\rho\wedge\omega^1 - \lambda\wedge\omega^2 &&\\
\mathrm{d}\omega^3 &= \quad\quad - \rho\wedge\omega^2 - \lambda\wedge\omega^3 &&
\end{aligned}$$

for some (also unique) functions A^i on B. Moreover, for any choice of connection forms α and β as above, one must have

$$\alpha - \lambda \equiv \beta - \rho \equiv 0 \mod \omega^1, \omega^2, \omega^3.$$

Computing the exterior derivatives of the structure equations yields formulae of the form

$$\begin{aligned}
\mathrm{d}A^1 &= A^1\,\lambda \qquad\qquad + A^1_1\,\omega^1 + A^1_2\,\omega^2 + A^1_3\,\omega^3\\
\mathrm{d}A^2 &= A^2\,\lambda - 2A^1\,\rho + A^2_1\,\omega^1 + A^2_2\,\omega^2 + A^2_3\,\omega^3\\
\mathrm{d}A^3 &= A^3\,\lambda - 2A^2\,\rho + A^3_1\,\omega^1 + A^3_2\,\omega^2 + A^3_3\,\omega^3,
\end{aligned}$$

while

$$\begin{aligned}
\mathrm{d}\lambda &= \rho\wedge(A^2\omega^2 - A^1\omega^3) + (A^1_1{+}A^2_2{+}A^3_3)\,\omega^2\wedge\omega^3 - K^1\,\omega^3\wedge\omega^1 - K^2\,\omega^1\wedge\omega^2\\
\mathrm{d}\rho &= \rho\wedge(A^2\omega^3 - A^3\omega^2) + K^1\,\omega^2\wedge\omega^3 + K^2\,\omega^3\wedge\omega^1 + K^3\,\omega^1\wedge\omega^2
\end{aligned}$$

for some unique functions A^i_j and K^i on B, and this shows that, except when the A^i vanish, λ and ρ are *not* invariant under the G-action on B and hence do not define a connection on B.

Still, because the forms λ and ρ are uniquely characterized by the structure equations, one has that the coframing $(\omega^1, \omega^2, \omega^3, \lambda, \rho)$ is a canonical coframing on B. Thus, the invariants of coframings satisfying structure equations of the above form will be invariants of the original G-structure B and *vice versa*. Cartan would have regarded this as the solution of the equivalence problem for these G-structures and would have used it to define the fundamental invariants of the G-structure.

Indeed, even though no connection has been defined, the structure equations derived so far can be used to construct tensorial invariants of the G-structure B. For example, one finds from the above structure equations that the three 1-forms

$$\begin{aligned}
\eta^1 &= A^1\,\omega^1\\
\eta^2 &= A^1\,\omega^2 - \tfrac12\,A^2\,\omega^1\\
\eta^3 &= A^1\,\omega^3 - \tfrac12\,A^2\,\omega^2 + \tfrac14\,A^3\,\omega^1
\end{aligned}$$

are the π-pullbacks of well-defined forms on M and that, on the open set U where $A^1 \neq 0$, they define a canonical coframing of U, which can be used to reduce the structure group. One also finds that $Q = \big((A^2)^2 - 2A^1A^3\big)\,(\omega^1)^2$ is the π-pullback of a well-defined rank 1 quadratic form on M.

3.2. **Covariants and derived covariants.** In the general theory of the method of equivalence, when one has defined a canonical coframing on a G-structure B (or, more generally, on some prolongation of the G-structure), then any equivalence of G-structures preserves the forms in the coframing and hence also preserves the functions that arise as coefficients of the exterior derivatives of these forms when they are expressed in terms of the coframing. These coefficient functions are known as the (generalized) *covariants* of the G-structure. Taking the exterior derivative of such a covariant and expressing it as a linear combination of the elements of the coframing then yields coefficient functions that are known as the *derived covariants* of the G-structure. In the general theory, one shows that, in an appropriate sense, the iterated derived covariants yield a complete set of invariants of G-structures.

In the present case, one can use the canonical coframing to generate the derived covariants of the G-structure by starting with the initial covariants I (such as, for example the A^i and K^j), defining

$$\mathrm{d}I = I_\lambda\,\lambda + I_\rho\,\rho + I_1\,\omega^1 + I_2\,\omega^2 + I_3\,\omega^3,$$

and adding the I_λ, I_ρ, and the I_i to the list of known covariants and then repeating the process. In this way, one generates *all* (in a suitable sense) of the differential invariants.

Of course, when iterating the derivations, one should keep in mind that, because $\mathrm{d}(\mathrm{d}I) = 0$, not all of the second derived covariants of a given covariant are independent. In fact, there will be 10 relations among these (potentially) 25 second derived invariants that allow one to eliminate 10 of them in favor of polynomials in lower order invariants.

For purposes of an argument to be presented below, let us say that a (derived) covariant I has λ-weight n if $I_\lambda = nI$. Thus, for example, the A^i all have λ-weight 1, the A^i_j and the K^i have λ-weight 2, and so on. Note that λ-weight is logarithmic in the sense that if $I_\lambda = nI$ and $J_\lambda = mJ$, then $(IJ)_\lambda = (m{+}n)\,IJ$. Moreover, if $I_\lambda = nI$ then the identity $\mathrm{d}(\mathrm{d}I) = 0$ implies that $(I_j)_\lambda = (n{+}1)\,I_j$. In particular, note that, in the polynomial ring $\mathcal{I}$ over $\mathbb{R}$ generated by the derived covariants, only linear combinations of the A^i have λ-weight equal to 1.

By computing the Cartan characters of the structure equations above (which, as it turns out, are involutive) one finds that $\mathcal{I}$ is freely generated as a polynomial ring over $\mathbb{R}$ by a set $S \subset \mathcal{I}$ of independent invariants that consists of $N_k = 2k^2{+}3k{-}2$ elements of differential order k (and λ-weight k) for each $k \geq 1$. For example, the elements A^i are the 3 elements of differential order 1, and the A^i_j and K^j are the 12 elements of differential order 2.

In particular, the necessary and sufficient condition that two G-structures B and B' agree up to diffeomorphism to order $k \geq 1$ at some frames $u \in B$ and $u' \in B'$ is that the $N_1 + \cdots + N_k$ algebraically independent elements of S of order at most k have the same values at u' as they do at u. Moreover, one can freely specify the values of these invariants at a frame $u \in B$.

3.3. **The nonexistence of a canonical connection.** I now want to explain why the above analysis shows that there cannot be any process that yields a canonical connection for the general G-structure on a 3-manifold.

Suppose that there did exist some process that produced a canonical connection for every G-structure. The connection forms α and β for this purported canonical connection would have to have the form

$$\begin{aligned}\alpha &= \lambda + I^1\,\omega^1 + I^2\,\omega^2 + I^3\,\omega^3\\ \beta &= \rho + J^1\,\omega^1 + J^2\,\omega^2 + J^3\,\omega^3\end{aligned}$$

for some functions I^i and J^i that are universal expressions in the derived covariants up to some finite order k. Moreover, these expressions would have to be smooth functions of the elements of S of order less than or equal to k that were also weighted homogeneous of λ-weight $+1$ since, otherwise the expressions $I^i\omega^i$ and $J^i\omega^j$ would not be weighted homogeneous of λ-weight 0, which is, of course, necessary. This implies that the I^i and the J^i must be polynomials in these elements of S that are weighted homogeneous of λ-weight $+1$, but, as we have seen, this can only happen if they are constant linear combinations of the A^i.

Now, expanding the condition that $\mathrm{d}\alpha$ and $\mathrm{d}\beta$ be 2-forms that are quadratic in the ω^i yields

$$\left.\begin{aligned} \mathrm{d}I^3 &\equiv I^3\,\lambda + A^1\,\rho \\ \mathrm{d}J^3 &\equiv J^3\,\lambda - A^2\,\rho \end{aligned}\right\} \quad \mathrm{mod}\ \omega^1, \omega^2, \omega^3.$$

Since I^3 and J^3 must be linear combinations of the A^i, this implies that $I^3 = c_1\,A^1 - \frac{1}{2}\,A^2$ while $J^3 = c_2 A^1 + \frac{1}{2}\,A^3$ for some constants c_1 and c_2. Substituting this information back into the formulae for $\mathrm{d}\alpha$ and $\mathrm{d}\beta$ and simplifying then yields

$$\left.\begin{aligned} \mathrm{d}I^2 &\equiv I^2\,\lambda + (c_1\,A^1 - \tfrac{3}{2}\,A^2)\,\rho \\ \mathrm{d}J^2 &\equiv J^2\,\lambda + (c_2\,A^1 + \tfrac{3}{2}\,A^3)\,\rho \end{aligned}\right\} \quad \mathrm{mod}\ \omega^1, \omega^2, \omega^3.$$

This implies that $I^2 = c_3\,A^1 - \frac{1}{2}c_1\,A^2 + \frac{3}{4}\,A^3$ for some constant c_3, but there is no linear combination of the A^i that satisfies the above equation for J^2.

Consequently, the necessary formulae for the I^i and J^i cannot be satisfied, and one concludes that there can be no local algorithm that yields a canonical connection for all G-structures on all 3-manifolds.

Remark 1. This does not mean that there is no process that yields a canonical connection for *some* G-structures. For example, for the G-structures that satisfy the open condition that $A^1\omega^1$ be nowhere vanishing, which is a well-defined condition on G-structures, one can always define a unique section of B as the image of the locus in B where $A^1 = 1$ and $A^2 = 0$. Now, one can choose the unique connection on B for which this 'canonical' section is parallel, and this defines a 'canonical' flat connection on B. The corresponding α and β in terms of λ and ρ in this case are found by solving the equations

$$\begin{aligned} A^1\alpha = \mathrm{d}A^1 &= A^1\,\lambda \qquad\qquad + A^1_1\,\omega^1 + A^1_2\,\omega^2 + A^1_3\,\omega^3 \\ A^2\alpha - A^1\beta = \mathrm{d}A^2 &= A^2\,\lambda - \ A^1\,\rho + A^2_1\,\omega^1 + A^2_2\,\omega^2 + A^2_3\,\omega^3 \end{aligned}$$

for α and β. Note, though, that this requires dividing by A^1, so that the resulting formulae for α and β will not, generally, be smooth for arbitrary G-structures for which the 1-form $A^1\omega^1$ can vanish.

References

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Duke University Mathematics Department, PO Box 90320, Durham, NC 27708-0320
E-mail address: bryant@math.duke.edu
URL: http://www.math.duke.edu/~bryant