The Problem. Compute the conjugate locus of $\mathrm{SL}(2,\mathbb{R})$ endowed with the 'natural' left-invariant Riemannian metric as described in my answer to the MathOverflow Question 108280

The geodesic formula: As explained there, the formula for the geodesic leaving $I_2 \in \mathrm{SL}(2,\mathbb{R})$ with velocity

$$v = \begin{pmatrix} v_1 & v_2 + v_3 \\ v_2 - v_3 & -v_1 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R})$$

is given by $\gamma_v(t) = e^{t\,v^T} e^{t\,(v - v^T)}$. Thus, the *geodesic* exponential mapping for this metric is

$$E(v) = e^{v^T} e^{(v - v^T)}.$$

(Here, 'v^T' denotes the transpose of v.) Meanwhile, since $v^2 = -\det(v)\, I_2$, it follows that the formula for the Lie group exponential of v is

$$e^v = c\big(\det(v)\big)\, I_2 + s\big(\det(v)\big)\, v$$

where c and s are the entire analytic functions defined on the real line that satisfy $c(t^2) = \cos(t)$ and $s(t^2) = \sin(t)/t$ (and hence satisfy $c(-t^2) = \cosh(t)$ and $s(-t^2) = \sinh(t)/t$). Note that, in particular, these functions satisfy the useful identities

$$c(y)^2 + y s(y)^2 = 1, \qquad c'(y) = -\tfrac{1}{2}\, s(y), \qquad \text{and} \qquad s'(y) = \big(c(y) - s(y)\big)/(2y).$$

Using this, the obvious identity $\det(v) = {v_3}^2 - {v_1}^2 - {v_2}^2$ and the above formulae, we can compute the pullback under E of the canonical left invariant form on $\mathrm{SL}(2,\mathbb{R})$ as follows.

$$E^*(g^{-1}\,\mathrm{d}g) = E(v)^{-1}\,\mathrm{d}\big(E(v)\big) = e^{-(v-v^T)}\left[e^{-v^T}\,\mathrm{d}(e^{v^T}) + \mathrm{d}(e^{(v-v^T)})\,e^{-(v-v^T)}\right] e^{(v-v^T)}.$$

Expanding this using the above formula for the Lie group exponential and setting

$$E^*(g^{-1}\,\mathrm{d}g) = \begin{pmatrix} \omega_1 & \omega_2 + \omega_3 \\ \omega_2 - \omega_3 & -\omega_1 \end{pmatrix},$$

we find, after setting $\det(v) = \delta$ for brevity, that

$$\omega_1 \wedge \omega_2 \wedge \omega_3 = s(\delta)\left(s(\delta) - 2({v_1}^2 + {v_2}^2)\frac{\big(c(\delta) - s(\delta)\big)}{\delta} \right) \mathrm{d}v_1 \wedge \mathrm{d}v_2 \wedge \mathrm{d}v_3\,.$$

(Note, by the way, that $\frac{c(\delta) - s(\delta)}{\delta}$ is an entire analytic function of δ.)

It follows that the degeneracy locus for the geodesic exponential map $E : \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ is the union of the locus described by the two equations

$$s\big(\det(v)\big) = 0 \tag{1}$$

and

$$s\Big(\det(v)\Big) - 2({v_1}^2{+}{v_2}^2)\frac{\Big(c\Big(\det(v)\Big) - s\Big(\det(v)\Big)\Big)}{\det(v)} = 0 \tag{2}$$

Now, $s(t) \geq 1$ when $t \leq 0$, while $s(t) = 0$ for $t > 0$ implies that $t = (k\pi)^2$ for some integer $k > 0$. Thus, the first locus is given by the hyperboloids

$$\det(v) = {v_3}^2 - {v_1}^2 - {v_2}^2 = k^2\pi^2, \quad k = 1, 2, \ldots$$

Meanwhile, when $t \leq 0$, the expression $\frac{c(t)-s(t)}{t}$ is strictly negative, while $s(t) \geq 1$, so it follows that the second locus has no points in the region $\det(v) \leq 0$, i.e., no geodesic γ_v with $\det(v) < 0$ has any conjugate points. Finally, a little elementary analytic geometry shows that the locus described by (2) is a countable union of surfaces Σ_k of revolution in $\mathfrak{sl}(2, \mathbb{R})$ that can be described in the form

$${v_3}^2 = ({v_1}^2 + {v_2}^2) + \Big(k + f_k({v_1}^2 + {v_2}^2)\Big)^2\pi^2, \qquad k = 1, 2, \ldots$$

where $f_k : [0, \infty) \to [0, \frac{1}{2})$ is a strictly increasing real-analytic function on $[0, \infty)$ that satisfies $f_k(0) = 0$. In particular, it follows that, for a $v \in \Sigma_k$, we have $k^2\pi^2 \leq \det(v) < (k + \frac{1}{2})^2\pi^2$.

Consequently, the *first* conjugate locus is the image under E of the hyperboloid $\det(v) = \pi^2$. Note that, by the above formulae, this image in $SL(2, \mathbb{R})$ is simply the subgroup $SO(2) \subset SL(2, \mathbb{R})$.