Some Remarks on Cartan's Structure Equations

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0. Introduction.

This is a set of notes on Cartan's generalization of the Third Fundamental Theorem of Lie. The reader will recall that this theorem of Lie states that if $T^i_{jk} = -T^i_{kj}$ are constants (where the index ranges are $1 \le i, j, k \le n$) that satisfy the *Jacobi conditions*

$$T^i_{mj}\,T^m_{kl} + T^i_{mk}\,T^m_{lj} + T^i_{ml}\,T^m_{jk} = 0 \tag{1}$$

(note the summation convention), then there exist linearly independent vector fields $\mathbf{X}_i$ on a neighborhood of the origin in $\mathbb{R}^n$ so that

$$[\mathbf{X}_i, \mathbf{X}_j] = T^k_{ij}\,\mathbf{X}_k\,. \tag{2}$$

Alternatively, the dual 1-forms ω^i should satisfy

$$d\omega^i = -\tfrac{1}{2}\,T^i_{jk}\,\omega^j \wedge \omega^k\,. \tag{3}$$

Taking the exterior derivative both sides of (3) and using the fact that $d^2 = 0$ shows that the conditions (1) are necessary for there to be linearly independent 1-forms ω^i satisfying (3). Lie's theorem asserts that these necessary conditions are sufficient for this existence result to hold. Moreover, Lie's existence theorem has a corresponding uniqueness theorem, which asserts that there is, up to local diffeomorphism, only one solution to (3).

Cartan generalized this theorem by considering the following problem: Suppose that one has putative structure equations of the following form:

$$d\omega^i = -\tfrac{1}{2}\,T^i_{jk}(a)\,\omega^j \wedge \omega^k, \tag{4}$$

where the summation convention is understood and

1. The 1-forms ω^i for $1 \le i \le n$ are supposed to be linearly independent,
2. The functions $a = (a^\alpha)$ for $1 \le \alpha \le s$ are supposed to be unknown,
3. The functions $T^i_{jk} = -T^i_{kj}$ are analytic on $\mathbb{R}^s$.

Then one wants to understand sufficient conditions, analogous to the Jacobi conditions, for such structure equations to have 'solutions'. Moreover, one wants to understand the 'generality' this space of 'solutions'.

Explaining Cartan's solution of this problem is the object of these notes.

1. Necessary conditions

The first task is to understand the obvious necessary conditions. These are obtained, just as in Lie's theorem, by taking the exterior derivatives of the equations in (4). One obtains the relations

$$
\begin{aligned}
0 = &-\frac{1}{2}\,\frac{\partial T^i_{jk}}{\partial a^\sigma}(a)\,da^\sigma\wedge\omega^j\wedge\omega^k \\
&+\frac{1}{6}\,\bigl(T^i_{mj}(a)\,T^m_{kl}(a)+T^i_{mk}(a)\,T^m_{lj}(a)+T^i_{ml}(a)\,T^m_{jk}(a)\bigr)\;\omega^j\wedge\omega^k\wedge\omega^l
\end{aligned}
\tag{5}
$$

Since, on a 'solution' to the above equations, there will exist functions b^σ_j

$$
da^\sigma = b^\sigma_j\,\omega^j\,. \tag{6}
$$

Now, substituting this relation into (5) yields

$$
T^i_{mj}(a)\,T^m_{kl}(a)+T^i_{mk}(a)\,T^m_{lj}(a)+T^i_{ml}(a)\,T^m_{jk}(a)=\frac{\partial T^i_{jk}}{\partial a^\sigma}(a)\,b^\sigma_l+\frac{\partial T^i_{kl}}{\partial a^\sigma}(a)\,b^\sigma_j+\frac{\partial T^i_{lj}}{\partial a^\sigma}(a)\,b^\sigma_k\,.
$$

Thus, a necessary condition is that this set of linear equations for the functions b^σ_i be solvable for at least some value of the variables $a=(a^\sigma)$. In the analytic category, the set of such values of a will be an open dense set in some analytic subvariety of a-space.

2. Sufficient Conditions

The necessary conditions derived above are far from sufficient, as simple examples show. Cartan proposed a sharper set of conditions that do turn out to be sufficient. Here they are:

Condition 1: There should exist analytic functions B^σ_i on $\mathbb{R}^s$ so that the equations

$$
T^i_{mj}\,T^m_{kl}+T^i_{mk}\,T^m_{lj}+T^i_{ml}\,T^m_{jk}=\frac{\partial T^i_{jk}}{\partial a^\alpha}\,B^\alpha_l+\frac{\partial T^i_{kl}}{\partial a^\alpha}\,B^\alpha_j+\frac{\partial T^i_{lj}}{\partial a^\alpha}\,B^\alpha_k \tag{7}
$$

hold identically on $\mathbb{R}^s$.

Condition 2: For every value of $a\in\mathbb{R}^s$, the subspace $A(a)\subset\mathbb{R}^s\otimes\Lambda^2\bigl((\mathbb{R}^n)^*\bigr)$ spanned by the elements

$$
A_\sigma(a)=\frac{\partial T^i_{jk}}{\partial a^\alpha}(a)\mathbf{e}_i\otimes\bigl(\mathbf{e}^j\wedge\mathbf{e}^k\bigr) \tag{8}
$$

should have dimension s and be involutive, with characters $s_2,\ldots,s_n$.

Remark: If V and W are a vector spaces of dimensions n and m, respectively, and $A \subset W \otimes \Lambda^2(V^*)$ is a subspace, then the notion of 'involutive' is defined as follows: For the generic flag of subspaces

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V,$$

let the dimension of the reduced space $A_k \subset W \otimes \Lambda^2(V_k^*)$ be $s_2 + \cdots + s_k$. (Of course, since $\Lambda^2(V_1^*) = 0$, it follows that $s_1 = 0$ by definition.) The (first) prolongation of A is the subspace $A^{(1)} \subset A \otimes V^*$ that is the kernel of the composition

$$A \otimes V^* \longrightarrow \big(W \otimes \Lambda^2(V^*)\big) \otimes V^* \longrightarrow W \otimes \Lambda^3(V^*).$$

The Cartan inequality in this case is

$$\dim A^{(1)} \leq 2\,s_2 + 3\,s_3 + \cdots + n\,s_n\,. \tag{9}$$

The subspace A is said to be *involutive* if equality holds in (9).

Cartan's result can now be stated approximately as follows:

Theorem: (Cartan) Suppose that the functions T^i_{jk} satisfy the two conditions listed above. Then there exist linearly independent 1-forms ω^i $(1 \leq i \leq n)$ on a neighborhood U of the origin in $\mathbb{R}^n$ and a real analytic mapping $a : U \to \mathbb{R}^s$ so that the equations (4) hold. Moreover, up to diffeomorphism, the general solution depends on s_2 functions of two variables, s_3 functions of three variables, ..., and s_n functions of n variables.

Remark: Note that, because the equations are invariant under diffeomorphism, one must be careful to count solutions only up to diffeomorphism. See below for a more precise statement about what this means.

3. An example.

Here is an example of an application of this method to a problem of interest. Consider the problem of determining the generality of local torsion-free $\mathrm{GL}(2,\mathbb{R}) \cdot \mathrm{GL}(m,\mathbb{R})$-structures on $\mathbb{R}^{2m}$.

As is easy to see, if $F \to U \subset \mathbb{R}^{2m}$ is a torsion-free $\mathrm{GL}(2,\mathbb{R}) \cdot \mathrm{GL}(m,\mathbb{R})$-structure on $U \subset \mathbb{R}^{2m}$, then there is a prolongation of F to a second order structure $F^{(1)}$ with structure group a semi-direct product of $\mathrm{GL}(2,\mathbb{R}) \cdot \mathrm{GL}(m,\mathbb{R})$ with $\mathbb{R}^{2m}$ on which there exists a Cartan connection with values in $\mathrm{SL}(m{+}2,\mathbb{R})$ with structure equations

$$\begin{aligned}
d\omega^\alpha_j &= -\phi^\alpha_\beta\wedge\omega^\beta_j - \omega^\alpha_i\wedge\psi^i_j\\
d\psi^i_j &= -\psi^i_k\wedge\psi^k_j - \eta^i_\beta\wedge\omega^\beta_j\\
d\phi^\alpha_\beta &= -\phi^\alpha_\gamma\wedge\phi^\gamma_\beta - \omega^\alpha_i\wedge\eta^i_\beta + F^\alpha_{\beta\gamma\delta}\,\omega^\gamma_1\wedge\omega^\delta_2\\
d\eta^i_\beta &= -\psi^i_j\wedge\eta^j_\beta - \eta^i_\alpha\wedge\phi^\alpha_\beta + G^i_{\beta\gamma\delta}\,\omega^\gamma_1\wedge\omega^\delta_2
\end{aligned}$$

$$\begin{aligned}
dF^\alpha_{\beta\gamma\delta} &= -F^\epsilon_{\beta\gamma\delta}\,\phi^\alpha_\epsilon + F^\alpha_{\epsilon\gamma\delta}\,\phi^\epsilon_\beta + F^\alpha_{\beta\epsilon\delta}\,\phi^\epsilon_\gamma + F^\alpha_{\beta\gamma\epsilon}\,\phi^\epsilon_\delta + R^{\alpha i}_{\beta\gamma\delta\epsilon}\,\omega^\epsilon_i\\
dG^i_{\beta\gamma\delta} &= -G^j_{\beta\gamma\delta}\,\psi^i_j + G^i_{\epsilon\gamma\delta}\,\phi^\epsilon_\beta + G^i_{\beta\epsilon\delta}\,\phi^\epsilon_\gamma + G^i_{\beta\gamma\epsilon}\,\phi^\epsilon_\delta - F^\alpha_{\beta\gamma\delta}\,\eta^i_\alpha + Q^{ij}_{\beta\gamma\delta\epsilon}\,\omega^\epsilon_j
\end{aligned}$$

The index ranges are understood to be $1\le i,j,k\le 2$ and $1\le\alpha,\beta,\gamma\le m$. The forms listed above are linearly independent except for the single trace relation $\psi^i_i+\phi^\alpha_\alpha=0$. The functions F, G, R, and Q must satisfy the relations

$$\begin{aligned}
F^\alpha_{\beta\gamma\delta} &= F^\alpha_{\gamma\beta\delta} = F^\alpha_{\beta\delta\gamma}, &\qquad F^\alpha_{\alpha\gamma\delta} &= 0,\\
G^i_{\beta\gamma\delta} &= G^i_{\gamma\beta\delta} = G^i_{\beta\delta\gamma}
\end{aligned}$$

as well as the relations

$$R^{\alpha i}_{\beta\gamma\delta\epsilon} = P^{\alpha i}_{\beta\gamma\delta\epsilon} + \frac{1}{m{+}3}\left(\delta^\alpha_\beta\,G^i_{\gamma\delta\epsilon} + \delta^\alpha_\gamma\,G^i_{\beta\delta\epsilon} + \delta^\alpha_\delta\,G^i_{\beta\gamma\epsilon} - (m{+}2)\delta^\alpha_\epsilon\,G^i_{\beta\gamma\delta}\right)$$

where $P^{\alpha i}_{\beta\gamma\delta\epsilon}$ is fully symmetric in its lower indices and satisfies $P^{\alpha i}_{\alpha\beta\gamma\delta}=0$. Finally, $Q^{ij}_{\beta\gamma\delta\epsilon}$ must be fully symmetric in its lower indices.

Note that, in the application of Cartan's theorem, the 1-forms play the role of the ω^i, the independent coefficients in F and G play the role of coordinates on A, while the independent coefficients in P and Q play the role of coordinates on $A^{(1)}$.

While it's true that the number n is $(m{+}2)^2-1=m^2{+}4m{+}3$, it's also clear from the structure equations that only the ω^α_i are effectively involved in the computation of the characters (since it is only these terms that appear with non-constant coefficients in the stucture equations). Thus, modulo what should be thought of as Cauchy characteristics, the effective dimension should be more like $n=2m$.

From the formulae above, one sees that Cartan's first condition is automatically satisfied. Using the symmetries of the coefficients, the dimensions

$$\dim A = (m+2)\binom{m+2}{3} - \binom{m+1}{2} = \frac{1}{6}\,m(m+1)(m^2+4m+1)$$

and

$$\dim A^{(1)} = (2m+4)\binom{m+3}{4} - 2\binom{m+2}{3} = \frac{1}{12}\,m(m+1)(m+2)(m^2+5m+2)$$

are easily computed. It remains to compute the characters, which turn out to be $s_k=(k-1)(m^2-(k-4)m-2k+3)$ for $2\le k\le m{+}1$, and $s_k=0$ for $k>m{+}1$. Thus, up to diffeomorphism, the general solution depends on $s_{m+1}=m(m{+}1)$ functions of $m{+}1$ variables.