NOTES ON EXTERIOR DIFFERENTIAL SYSTEMS

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ABSTRACT. These are notes for a very rapid introduction to the basics of exterior differential systems and their connection with what is now known as Lie theory, together with some typical and not-so-typical applications to illustrate their use.

CONTENTS

1. Introduction

Around the beginning of the 20th century, Élie Cartan developed a theory of partial differential equations that was well-suited for the study of local problems in differential geometry. His fundamental insight was that many local geometric problems could be recast as problems in which one seeks to find or to classify a collection of functions and differential forms satisfying some given 'structure equations', i.e., conditions on the exterior derivatives of the given functions and forms. The invariance of the exterior derivative under diffeomorphism thus made possible an approach to differential equations arising in geometry that Cartan then developed and applied in a very large number of situations, beginning with his theory of 'infinite groups' (what we now call pseudo-groups) and continuing throughout his later work in classical differential geometry.

In Cartan's formulation, he was mainly concerned with systems of equations that were generated by functions, 1-forms, and their exterior derivatives, and Cartan's collection of techniques is now generally called the theory of 'exterior differential systems', or simply 'EDS'. Later, Erich Kähler [10] showed that Cartan's theory could be usefully extended to systems generated by differential forms of arbitrary degree, and the resulting extension is now known as Cartan-Kähler Theory.

Cartan and his students used EDS with great success in situations in which a given problem could be recast as a system satisfying the hypothesis of *involutivity*, which many important systems did. For non-involutive systems, Cartan introduced the notion of *prolongation* as a process whose purpose is to replace a given exterior differential system with one that has essentially the same solutions but is also involutive, the hope being that all of the solutions of any system would be describable as the solutions of involutive systems. Cartan was never able to prove that prolongation succeeded in doing this in all cases, but, later, Masatake Kuranishi [11] did succeed in proving a version of the desired prolongation theorem, one that applies in nearly all cases of interest, thus essentially completing Cartan's theory on an important point.

In these notes, I introduce the basics of exterior differential systems, and cover the essential definitions and theorems. I do not attempt to discuss the proofs of fundamental results such as Cartan's Bound (aka Cartan's Test), the Cartan-Kähler Theorem, or the Cartan-Kuranishi Theorem. For proofs of these, the reader can consult any of the standard sources in the subject, such as [1]. Instead, I concentrate on producing examples that illustrate the ideas and techniques of EDS.

My choice of subject matter and examples was greatly influenced by the audience of the workshop at which this material was written, and I have made no attempt to give a more comprehensive view of the standard topics in exterior differential systems. Rather, I have focused on applications to 'Cartan structure equations' associated to differential geometric problems and tried to show how Cartan's theory is connected with modern-day Lie theory.

1.1. **Differential ideals.** Let M^n be a smooth n-manifold. An *exterior differential system* on M is a graded, differentially closed ideal $\mathcal{I} \subset \mathcal{A}^*(M)$.

While it is not strictly necessary, it simplifies some statements if one assumes that $\mathcal{I}$ is generated in positive degrees, i.e., $\mathcal{I}^0 = \mathcal{I} \cap \mathcal{A}^0(M) = (0)$, so I will assume this throughout the notes.

1.2. **Integral manifolds and elements.** An *integral manifold* of $\mathcal{I}$ is a submanifold $f : N \to M$ such that $f^*(\phi) = 0$ for all $\phi \in \mathcal{I}$.

Remark 1. In most applications of exterior differential systems, the integral manifolds of a certain dimension (often the maximal dimension) of a given differential ideal $\mathcal{I}$ represent the local solutions of some geometric problem that can be expressed in terms of partial differential equations. Thus, one is interested in techniques for describing the integral manifolds of a given $\mathcal{I}$.

An *integral element* of $\mathcal{I}$ is a p-plane $E \in \mathrm{Gr}_p(TM)$ such that $\iota_E^*(\phi) = 0$ for all $\phi \in \mathcal{I}$. The set of p-dimensional integral elements of $\mathcal{I}$ is a closed subset $\mathcal{V}_p(\mathcal{I}) \subseteq \mathrm{Gr}_p(TM)$. It is not always a smooth submanifold of this bundle.

Remark 2. Every tangent plane to an integral manifold of $\mathcal{I}$ is an integral element of $\mathcal{I}$. The fundamental problem in exterior differential systems is to decide, for a given $E \in \mathcal{V}_p(\mathcal{I})$, whether there is an integral manifold of $\mathcal{I}$ that has E as one of its tangent spaces.

1.3. **Polar spaces.** While determining the structure of $\mathcal{V}_p(\mathcal{I})$ can be challenging, the problem of describing the $(p+1)$-dimensional integral elements that contain a given p-dimensional integral element is essentially a linear one.

Fix $E \in \mathcal{V}_p(\mathcal{I})$, with $E \subset T_xM$, and let $e_1, \ldots, e_p$ be a basis of E. The *polar space* (sometimes called the *enlargement space*) of E is the subspace

$$H(E) = \{ v \in T_xM \mid \phi(v, e_1, \ldots, e_p) = 0 \; \forall \phi \in \mathcal{I}^{p+1} \} \subset T_xM.$$

From its definition, any $E_+ \in \mathcal{V}_{p+1}(\mathcal{I})$ that contains E must be contained in $H(E)$ and, conversely, any $E_+ \in \mathrm{Gr}_{p+1}\big(H(E)\big)$ that contains E belongs to $\mathcal{V}_{p+1}(\mathcal{I})$. Set $c(E) = \dim\big(T_xM/H(E)\big)$.

1.4. **Cartan's Bound and characters.** Let $E \in \mathcal{V}_n(\mathcal{I})$ be fixed, and let $F = (E_0, E_1, \ldots, E_{n-1})$ be a flag of subspaces of E, with $\dim E_i = i$. Thus,

$$(0)_x = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E \subset T_xM.$$

Note that E_i belongs to $\mathcal{V}_i(\mathcal{I})$. The following result is due to Cartan and Kähler.

Proposition 1 (Cartan's Bound). *Given $E \in \mathcal{V}_n(\mathcal{I})$ and a flag $F = (E_i)$ in E as above, there is an open E-neighborhood $U \subset \mathrm{Gr}_n(TM)$ such that $\mathcal{V}_n(\mathcal{I}) \cap U$ is contained in a smooth submanifold of U of codimension*

$$c(F) = c(E_0) + c(E_1) + \cdots + c(E_{n-1}).$$

Definition 1 (Regularity and Ordinarity). If $E \in \mathcal{V}_n(\mathcal{I})$ has a flag $F = (E_i)$ and an open neighborhood $U \subset \mathrm{Gr}_n(TM)$ such that $U \cap \mathcal{V}_n(\mathcal{I})$ is a smooth submanifold of U with codimension $c(F)$, one says that F is *regular* and that E is *ordinary*.[1]

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[1]N.B.: When comparing other sources, the reader should be aware that what I call 'ordinary' in these notes is usually called 'Cartan-ordinary', which should not be confused with the weaker notion 'Kähler-ordinary', which will not be needed in these notes (so I will not discuss it).

Let $\mathcal{V}_n^o(\mathcal{I}) \subset \mathcal{V}_n(\mathcal{I})$ denote the subset consisting of ordinary integral elements of $\mathcal{I}$. It is an open (possibly empty) subset of $\mathcal{V}_n(\mathcal{I})$ that is a smooth submanifold of $\mathrm{Gr}_n(TM)$, and the basepoint projection $\pi : \mathcal{V}_n^o(\mathcal{I}) \to M$ is a submersion. (This uses my standing assumption that $\mathcal{I}^0 = (0)$.)

Remark 3 (Characters and Cartan's Test). A 'dual' version of Cartan's Bound (often called 'Cartan's Test') is useful: For any $E \in \mathcal{V}_n(\mathcal{I})$ and any flag $F = (E_0, E_1, \ldots, E_{n-1})$, the *character sequence* of F is the sequence of nonnegative integers $\big(s_0(F), s_1(F), \ldots, s_n(F)\big)$ such that

$$s_i(F) = \begin{cases} c(E_0) & i = 0, \\ c(E_i) - c(E_{i-1}) & 1 \leq i < n, \\ \dim H(E_{n-1}) - n & i = n. \end{cases}$$

Then Cartan's Bound says that, near E, the subset $\mathcal{V}_n(\mathcal{I})$ is contained in a submanifold of $\mathrm{Gr}_n(TM)$ of dimension

$$\dim M + s_1(F) + 2s_2(F) + \cdots + ns_n(F).$$

Moreover, if, near E, the subset $\mathcal{V}_n(\mathcal{I})$ *is* a submanifold of $\mathrm{Gr}_n(TM)$ of this dimension, then E is ordinary and the flag F is regular.

When E is ordinary, the character sequence $\big(s_k(F)\big)$ is the same for all regular flags $F = (E_0, E_1, \ldots, E_{n-1})$ in E. This common sequence is known as the sequence of *Cartan characters* of E and simply written as the sequence $\big(s_k(E)\big)$. Moreover, the characters s_k are constant on the connected components of $\mathcal{V}_n^o(\mathcal{I})$.

2. Cartan-Kähler Theory

2.1. **A form of the Cartan-Kähler Theorem.** The main result needed in these notes is the following version of the Cartan-Kähler Theorem.[2]

Theorem 1 (Cartan-Kähler). *Suppose that $\mathcal{I}$ is a real-analytic exterior differential system on M that is generated in positive degree and that $E \in \mathcal{V}_n(\mathcal{I})$ is ordinary. Then there exists a real-analytic integral manifold of $\mathcal{I}$ that has E as one of its tangent spaces.*

Remark 4 (Generality). The Cartan-Kähler theorem constructs the desired integral manifold by solving a sequence of initial value problems via the Cauchy-Kowalevski Theorem. At each step in the sequence, one gets to choose appropriate initial data that determine the resulting integral manifold. Examining the proof of the Cartan-Kähler theorem, one finds that there is an open E-neighborhood $U \subset \mathcal{V}_n(\mathcal{I})$ such that the initial data that determines a connected integral manifold of $\mathcal{I}$ whose tangent spaces belong to U consists of $s_0(E)$ constants, $s_1(E)$ functions of 1 variable, $s_2(E)$ functions of 2 variables, ..., and $s_n(E)$ functions of n variables that are freely specifiable (i.e. 'arbitrary'), subject only to some open conditions.

Moreover, in the real-analytic setting, when the underlying manifold M is connected and the ideal $\mathcal{I}$ is generated in positive degree, the functions s_i are constant on the (necessarily smooth) submanifold $\mathcal{V}_n^o(\mathcal{I}) \subset \mathcal{V}_n(\mathcal{I})$ consisting of the ordinary integral elements of dimension n, and so one usually writes s_i instead of $s_i(E)$ (with n being understood, usually taken to be the maximal integer for which $\mathcal{V}_n^o(\mathcal{I})$

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[2]The standard, stronger version of the Cartan-Kähler Theorem has more technical hypotheses and so takes a bit longer to state. I won't need the stronger version in these notes.

is not empty). In this case, (as we will see in §4.3) the set $J_k^o(\mathcal{I})$ consisting of k-jets of ordinary integral manifolds (i.e., the ones whose tangent spaces are ordinary) is naturally a smooth manifold of dimension

$$N_k = n + s_0 + (k{+}1)\, s_1 + \binom{k+2}{2} s_2 + \cdots + \binom{k+n}{n} s_n\,.$$

Moreover, the natural projections $\pi_k^{k+1} : J_{k+1}^o(\mathcal{I}) \to J_k^o(\mathcal{I})$ are smooth submersions, surjective when $k \geq 1$. (Note that $J_1^o(\mathcal{I}) = \mathcal{V}_n^o(\mathcal{I})$.)

Thus, one often says that the ordinary integral manifolds of $\mathcal{I}$ 'depend on s_0 constants, s_1 functions of 1 variable, s_2 functions of 2 variables, ..., and s_n functions of n variables'.

Remark 5 (Significance of the last nonzero character). One sometimes encounters statements such as "only the last nonzero character really matters," which the writer usually phrases as something like "the solution depends on s_q functions of q variables" (where $s_q > 0$ and $s_k = 0$ for all $k > q$), thus ignoring all of the s_i for $i < q$.

The reason for this is that there is usually more than one way to describe the local solutions of a given geometric problem as the ordinary integral manifolds of some exterior differential system $\mathcal{I}$. Two such descriptions might well have different character sequences (some examples will be given below), but they always have the same last nonzero character (at the same level q).

Nevertheless, for any *given* exterior differential system $\mathcal{I}$, *its* full character sequence does have intrinsic meaning.

2.2. **Involutive tableau.** Let V and W be vector spaces over $\mathbb{R}$ of dimensions n and m, respectively, and let $A \subset W \otimes V^*$ be an r-dimensional linear subspace of the linear maps from V to W.[3] One wants to understand the space of maps $f : V \to W$ with the property that $f'(x)$ lies in A for all $x \in V$. Thus, f is being required to satisfy a set of homogeneous, constant coefficient, linear, first-order partial differential equations, a very basic system of PDE.

Set up an exterior differential system as follows: Let $M = V \times W \times A$ and let $x : M \to V$, $u : M \to W$, and $p : M \to A$ denote the projections. Let $\mathcal{I}_A$ be the ideal generated by the components of the W-valued 1-form $\theta = \mathrm{d}u - p\,\mathrm{d}x$. Thus, $\mathcal{I}_A$ is generated in degree 1 by $m = \dim W$ 1-forms and in degree 2 by the (at most) m independent 2-forms that are the components of $\mathrm{d}\theta = -\mathrm{d}p{\wedge}\mathrm{d}x$.

An n-plane $E \in \mathrm{Gr}_n(TM)$ at $(x_0, u_0, p_0) \in M$ for which $\mathrm{d}x : E \to V$ is an isomorphism will be described by equations of the form

$$\mathrm{d}u - q(E)\,\mathrm{d}x = \mathrm{d}p - s(E)\,\mathrm{d}x = 0$$

where $q(E)$ belongs to $W \otimes V^*$ and $s(E)$ belongs to $A \otimes V^* \subset (W \otimes V^*) \otimes V^*$. It will be an integral element of $\mathcal{I}_A$ if and only if, first $q(E) = p_0$, and, second $\big(s(E)\,\mathrm{d}x\big){\wedge}\mathrm{d}x = 0$. This last condition is equivalent to requiring that $s(E)$ lie in the intersection

$$A^{(1)} = (A \otimes V^*) \cap \big(W \otimes S^2(V^*)\big).$$

Let $r^{(1)}$ denote the dimension of this space. Thus, in the open subset $U \subset \mathrm{Gr}_n(TM)$ consisting of the n-planes E on which $\mathrm{d}x : E \to V$ is an isomorphism, $\mathcal{V}_n(\mathcal{I}_A)$ is

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[3]In the literature, A is often called a *tableau*, which is simply a borrowing from the French of the word used to describe a subspace of linear maps written out as a matrix whose entries satisfy some given linear relations.

defined by the conditions $q(E) = p$ and $s(E) \in A^{(1)}$. Hence, the space $\mathcal{V}_n(\mathcal{I}_A) \cap U$ is a submanifold of $\mathrm{Gr}_n(TM)$ of codimension $mn + (rn - r^{(1)})$.

Now let $(0) = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$ be a flag in V, and, for each k, let $A_k \subset W \otimes V_k^*$ denote the image of A under the projection $W \otimes V^* \to W \otimes V_k^*$. This defines a flag $F = (E_0, E_1, \ldots, E_{n-1})$ in any $E \in \mathcal{V}_n(\mathcal{I}_A)$ for which $\mathrm{d}x : E \to V$ is an isomorphism by letting $\mathrm{d}x(E_i) = V_i$. Inspection now shows that $c(E_i) = m + \dim A_i$, so Cartan's Bound becomes

$$mn + (rn - r^{(1)}) \geq c(E_0) + \cdots + c(E_{n-1}) = mn + \sum_{i=1}^{n-1} \dim A_i \,,$$

which, after rearrangement, becomes

$$\dim A^{(1)} = r^{(1)} \leq n \dim A - \sum_{i=1}^{n-1} \dim A_i \,.$$

In particular, whether an integral element on which $\mathrm{d}x$ is independent has a regular flag (and hence is ordinary) depends only on the subspace $A \subset W \otimes V^*$.

The numbers $s_i(F, A) = \dim A_i - \dim A_{i-1}$ for $1 \leq i \leq n$ are called the *characters* of the flag F with respect to A. In terms of the characters $s_i(F, A)$, the above inequality takes the form known as *Cartan's Test*, namely,

$$\dim A^{(1)} \leq s_1(F, A) + 2\, s_2(F, A) + \cdots + n\, s_n(F, A),$$

and equality holds iff F is a regular flag and the integral elements $E \in \mathcal{V}_n(\mathcal{I}_A)$ on which $\mathrm{d}x : E \to V$ is a isomorphism are ordinary. When such a flag F exists, the tableau A is said to be *involutive*, and the *Cartan characters* of A are $s_i(A) = s_i(F, A)$ (computed with respect to any regular flag F).

When A is involutive, the Cartan-Kähler theorem implies that the real-analytic integral manifolds of $\mathcal{I}_A$ exist and depend on $s_0 = m$ constants, $s_1(A)$ functions of 1 variable, $s_2(A)$ functions of 2 variables, etc.

In particular, if one takes the Taylor series of the 'general' solution $f : V \to W$ of the equations forcing $f'(x)$ to lie in A for all x, one gets

$$f(x) = f_0 + f_1(x) + f_2(x) + \cdots + f_k(x) + \cdots$$

where f_k is a W-valued homogeneous polynomial of degree k on V and hence lies in the subspace

$$A^{(k-1)} = \left(W \otimes S^k(V^*)\right) \cap \left(A \otimes S^{k-1}(V^*)\right),$$

which has dimension

$$\dim A^{(k-1)} = \sum_{j=1}^{n} \binom{j+k-2}{k-1} s_j(A) \,,$$

which is exactly what one would expect if f were to be thought of as being comprised of $s_0 = m$ constants, $s_1(A)$ functions of 1 variable, $s_2(A)$ functions of 2 variables, etc.

2.3. Linearization at an ordinary integral element.
The concept of involutivity turns out to be fundamental, so it is worthwhile to examine how this is connected with the notion of an ordinary integral element in general.

Thus, fix $E \in \mathcal{V}_n(\mathcal{I})$, with $E \subset T_x M$. Let $E^\perp \subset T_x^* M$ be the space of forms that vanish on E, and note that the ideal $(E^\perp) \subset \Lambda(T_x^* M)$ generated by $E^\perp$ consists of

the forms that vanish on $E \subset T_x M$. Let $\mathcal{I}_x \subset \Lambda(T_x^* M)$ denote the set of values of forms in $\mathcal{I}$ at the point x, Then $\mathcal{I}_x$ is contained in $(E^\perp)$ because E is an integral element of $\mathcal{I}$. Consider the quotient

$$I_E = (\mathcal{I}_x + (E^\perp)^2)/(E^\perp)^2 \subseteq (E^\perp)/(E^\perp)^2 \simeq E^\perp \otimes \Lambda(E^*).$$

This $I_E \subset E^\perp \otimes \Lambda(E^*)$ should be thought of as the 'linearization' of the ideal $\mathcal{I}$ at E. It generates an ideal (I_E) in the space of forms on $T_x M/E \oplus E$ (whose dual space is $E^\perp \oplus E^*$) that has $0 \oplus E \subset T_x M/E \oplus E$ as an integral element.

If E is, in addition, ordinary, then it is not difficult to show that $0 \oplus E$ is an ordinary integral element of (I_E), that a flag of the form $0 \oplus E_i$ is regular for $0 \oplus E$ if and only if $F = (E_i)$ is a regular flag for E, and that one has equality of characters $s_i(0 \oplus E) = s_i(E)$.

2.4. **Higher tableau.** This motivates the following: Given two vector spaces W and V (of dimensions m, and n, respectively), a graded[4] subspace $I \subset W^* \otimes \Lambda(V^*)$ is *involutive* if $0 \oplus V \subset W \oplus V$ is an ordinary integral element of the ideal $(I) \subseteq \Lambda\big((W \oplus V)^*\big)$ generated by I. In this case, set $s_i(I) = s_i(0 \oplus V)$ and let

$$A_I = \big\{ f \in \mathrm{Hom}(V, W) \mid \Gamma_f \in \mathcal{V}_n\big((I)\big) \big\} \subset W \otimes V^*,$$

where $\Gamma_f = \{(f(x), x) \mid x \in V \} \subset W \oplus V$; the subspace A_I is said to be the *tableau* of I. When I is involutive, A_I is also involutive and its characters are

$$s_i(A_I) = s_i(I) + s_{i+1}(I) + \cdots + s_n(I).$$

3. First Examples and Applications

I will now give a basic set of examples illustrating the concepts and applications of the Cartan-Kähler Theorem. Some are intended just to help the reader gain familiarity with the concepts, while others will turn out to have significant applications.

Example 1 (The Frobenius theorem). Suppose that $\mathcal{I}$ on M^{n+s} can be locally generated *algebraically* by s linearly independent 1-forms $\theta^1, \ldots, \theta^s$. In particular, since $\mathcal{I}$ is differentially closed, it follows that there are (local) 1-forms ϕ_b^a such that $\mathrm{d}\theta^a = \phi_b^a \wedge \theta^b$.

In particular, there is a unique n-dimensional integral element at each point $x \in M$, namely the n-dimensional subspace $E_x \subset T_x M$ on which each of the θ^a vanish. Thus, $\mathcal{V}_n(\mathcal{I}) \subset \mathrm{Gr}_n(TM)$ is simply a copy of M, in fact, the image of a smooth section of the bundle $\mathrm{Gr}_n(TM)$, so it is a smooth manifold of dimension $n+s$. Meanwhile, for any flag $F = (E_0, \ldots, E_{n-1})$ in E_x, one has $H(E_p) = E_x$, so $c(E_i) = s$ for $0 \le i < n$. In particular, $s_0(F) = s$ and $s_i(F) = 0$ for $0 < i < n$. Since $\dim \mathcal{V}_n(\mathcal{I}) = n+s = \dim M + s_1 + 2s_2 + \cdots + ns_n$, it follows that Cartan's Bound is saturated, and all of the elements of $\mathcal{V}_n(\mathcal{I})$ are ordinary and all their flags are regular.

By the Cartan-Kähler Theorem, every E_x is tangent to an integral manifold of $\mathcal{I}$ and the local integral manifolds near E depend on $s_0 = s$ constants.

Now, in this particular case, there is another way to get the same result, which is to use the Frobenius Theorem (which is even better since it applies in the smooth

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[4]This means that I is the direct sum of its subspaces $I^q = I \cap \big(W^* \otimes \Lambda^q(V^*)\big)$.

setting). This Theorem says that, locally, it is possible to choose closed generators $\theta^a = \mathrm{d}y^a$ for some functions $y^1, \ldots, y^s$ that form part of a coordinate system $x^1, \ldots, x^n, y^1, \ldots, y^s$. Then the local n-dimensional integral manifolds of $\mathcal{I}$ are the leaves defined by holding the y^a constant, so that the 'general' local n-dimensional integral manifold depends on s constants, in agreement with the prediction of the Cartan-Kähler Theorem.

Example 2 (A non-ordinary integral element). Let $M = \mathbb{R}^3$, with coordinates x, y, z, and let $\mathcal{I}$ be generated by the 2-forms $\mathrm{d}x \wedge \mathrm{d}z$ and $\mathrm{d}y \wedge \mathrm{d}z$. Then the 2-plane field defined by $\mathrm{d}z = 0$ consists of 2-dimensional integral elements, and these are the only 2-dimensional integral elements, so that $\mathcal{V}_2(\mathcal{I})$ is a smooth 3-manifold in $\mathrm{Gr}_2(TM)$. Since $\mathcal{I}^1 = 0$, one has $c(E_0) = 0$ for all E_0. Letting E_1 be spanned by $a\partial_x + b\partial_y$, where $(a,b) \neq 0$, one finds that $H(E_1)$ has dimension 2 and is defined by $\mathrm{d}z = 0$, so $c(E_1) = 1$. Thus, $c(E_0) + c(E_1) = 1$ while the codimension of $\mathcal{V}_2(\mathcal{I})$ in $\mathrm{Gr}_2(TM)$ is 2. Thus, $E_2 = H(E_1)$ has no regular flag and hence is not ordinary.

It may seem disappointing that the Cartan-Kähler Theorem does not apply to prove the existence of 2-dimensional integral manifolds, especially, since there evidently does exist an integral manifold tangent to every 2-dimensional integral element of $\mathcal{I}$, namely, the plane $z = z_0$.

To see why one should not expect Cartan-Kähler to apply in this case, consider a modification of this example got by instead considering the ideal $\mathcal{I}'$ generated by $\mathrm{d}x \wedge \mathrm{d}z$ and $\mathrm{d}y \wedge (\mathrm{d}z - y\,\mathrm{d}x)$. The ideals $\mathcal{I}$ and $\mathcal{I}'$ are algebraically equivalent at each point, so there is a unique 2-dimensional integral element of $\mathcal{I}'$ through each point, namely the 2-plane that satisfies $\mathrm{d}z - y\,\mathrm{d}x = 0$. Since the dimensions of the polar spaces are the same for $\mathcal{I}'$ as they are for $\mathcal{I}$, these integral elements of $\mathcal{I}'$ are not ordinary, and this is just as well because there evidently are not any integral surfaces of the equation $\mathrm{d}z - y\,\mathrm{d}x = 0$.

Example 3 (Lagrangian submanifolds). Let $M = \mathbb{R}^{2n}$ and let $\mathcal{I}$ be generated by the symplectic form

$$\Omega = \mathrm{d}p_1 \wedge \mathrm{d}x^1 + \cdots + \mathrm{d}p_n \wedge \mathrm{d}x^n.$$

Then the n-plane E spanned by the ∂_{x^i} is an integral element of $\mathcal{I}$, and, if one takes the flag $F = (E_0, \ldots, E_{n-1})$ so that E_i is spanned by the ∂_{x^j} with $1 \leq j \leq i$, then one computes that, for $0 < i < n$, the polar space $H(E_i)$ is the subspace defined by $\mathrm{d}p_1 = \mathrm{d}p_2 = \cdots = \mathrm{d}p_i = 0$. Thus, $c(E_i) = i$.

By Cartan's Bound, $\mathcal{V}_n(\mathcal{I})$ has codimension at least

$$C = 1 + 2 + \cdots + (n-1) = \tfrac{1}{2}n(n-1)$$

in $\mathrm{Gr}_n(TM)$ near E. Meanwhile, any $\tilde{E} \in \mathrm{Gr}_n(TM)$ on which the $\mathrm{d}x^i$ are linearly independent will be defined by unique equations of the form

$$\mathrm{d}p_i - s_{ij}(\tilde{E})\,\mathrm{d}x^i = 0$$

for some numbers $s_{ij}(\tilde{E})$, and these functions s_{ij}, together with the x^i and the p_i define a local coordinate system on an open subset of $\mathrm{Gr}_n(TM)$ that contains E (which is defined by $s_{ij}(E) = 0$).

By Cartan's Lemma, such an $\tilde{E}$ will be an integral element of $\mathcal{I}$ if and only if $s_{ij}(\tilde{E}) - s_{ji}(\tilde{E}) = 0$. This is $\tfrac{1}{2}n(n-1)$ independent equations on $\tilde{E}$, so that $\mathcal{V}_n(\mathcal{I})$ has codimension $\tfrac{1}{2}n(n-1) = C$ in $\mathrm{Gr}_n(TM)$ near E. Consequently, E is ordinary, and the flag F is regular.

Of course, one already knows that Lagrangian manifolds exist, so this is not a surprise. Note, however, that what the Cartan-Kähler theorem would say is that one can specify an integral manifold on which the x^i are independent uniquely by choosing p_n to be an arbitrary function of the x^i, then choosing p_{n-1} subject to the condition that its partial in the x^n-direction equals the partial of p_n in the x^{n-1} direction (which determines p_{n-1} up to the addition of a function of $x^1, \ldots x^{n-1}$), then choosing p_{n-2} subject to the conditions that its partials in the x^n- and x^{n-1}-directions are determined by those of p_n and p_{n-1} (which determines p_{n-2} up to the addition of a function of $x^1, \ldots x^{n-2}$), etc. Thus, the integral manifolds are described by a choice of $1 = s_n(E)$ function of n variables, $1 = s_{n-1}(E)$ function of $n-1$ variables, etc., in agreement with the general theory.

Now, one can also specify a Lagrangian using only one function of n variables simply by taking $p_i = \frac{\partial u}{\partial x^i}$ for some function u of $x^1, \ldots, x^n$. However, for general $\mathcal{I}$, one cannot find such a formula that combines the 'arbitrary functions' in the general ordinary integral manifolds of $\mathcal{I}$ in this way.

Another way to interpret this 'discrepancy' is to note that the Lagrangian manifolds on which $dx^1 \wedge \cdots \wedge dx^n \neq 0$ are, by the above formula, put in correspondence with the arbitrary local function u of n variables, which, via its graph $\left(x^1, \ldots, x^n, u(x^1, \ldots, x^n)\right)$, is seen to be an integral manifold of the trivial ideal $\mathcal{I} = (0)$ on $\mathbb{R}^{n+1}$, which has Cartan characters

$$(s_0, s_1, \ldots, s_{n-1}, s_n) = (0, 0, \ldots, 0, 1).$$

Thus, this provides an example of the phenomenon that I mentioned earlier of two different exterior differential systems describing (local) solutions to the same problem. Note that they have the same last nonzero character, namely, $s_n = 1$, while their lower characters are not the same.

3.1. **A generalization of Lie's Third Theorem.** I now turn to a more substantial application of the Cartan-Kähler Theorem. In fact, it was one of the original motivations for Cartan to prove his original existence theorem in the first place. This will be a generalization of Lie's Third Fundamental Theorem, which asserts that every finite-dimensional Lie algebra is the Lie algebra of a (local) Lie transformation group.

Recall that, if G is a Lie group of dimension n and ω^i for $1 \leq i \leq n$ are a basis for the left-invariant 1-forms on G, then one has equations of the form

(3.1) $$d\omega^i = -\tfrac{1}{2} c^i_{jk}\, \omega^j \wedge \omega^k$$

for some unique constants $c^i_{jk} = -c^i_{kj}$. Since one has

$$0 = d(d\omega^i) = \tfrac{1}{6}\left(c^i_{mj}c^m_{kl} + c^i_{mk}c^m_{lj} + c^i_{ml}c^m_{jk}\right) \omega^j \wedge \omega^k \wedge \omega^l,$$

the constants c^i_{jk} satisfy the well-known quadratic equations

$$c^i_{mj}c^m_{kl} + c^i_{mk}c^m_{lj} + c^i_{ml}c^m_{jk} = 0$$

sometimes known as the *Jacobi identities*. Lie proved a converse to this result (now known as *Lie's Third Fundamental Theorem*), namely that, if a set of constants $c^i_{jk} = -c^i_{kj}$ with $1 \leq i, j, k \leq n$ satisfy the above Jacobi identities, then there exists a basis of left-invariant 1-forms ω^i on $\mathbb{R}^n$ that satisfy (3.1).

For many applications in differential geometry, Cartan needed a generalization of Lie's existence result, which he formulated as follows: Suppose that $C^i_{jk} = -C^i_{kj}$

and F_i^α (with $1 \le i, j, k \le n$ and $1 \le \alpha \le s$) are given functions on $\mathbb{R}^s$, and one wants to know whether or not there exist linearly independent 1-forms ω^i on $\mathbb{R}^n$ and a function $a = (a^\alpha) : \mathbb{R}^n \to \mathbb{R}^s$ that satisfy the *Cartan structure equations*

$$(3.2) \qquad \mathrm{d}\omega^i = -\tfrac{1}{2} C_{jk}^i(a)\, \omega^j \wedge \omega^k \qquad \text{and} \qquad \mathrm{d}a^\alpha = F_i^\alpha(a)\, \omega^i.$$

Such a pair (a, ω) will be said to be an *augmented coframing* satisfying the structure equations (3.2).

Applying the fundamental identity $\mathrm{d}^2 = 0$ yields necessary conditions in order for such a pair (a, ω) to exist: One must have $\mathrm{d}(C_{jk}^i(a)\, \omega^j \wedge \omega^k) = \mathrm{d}(\mathrm{d}\omega^i) = 0$ and $\mathrm{d}(F_i^\alpha(a)\, \omega^i) = \mathrm{d}(\mathrm{d}a^\alpha) = 0$. Expanding these identities using (3.2) and the assumed independence of the ω^i then yields that, if, for each $u_0 \in \mathbb{R}^s$, an augmented coframing (a, ω) on some n-manifold M exists satisfying (3.2) with $a(x) = u_0$ for some $x \in M$, then one must have

$$(3.3) \qquad F_j^\alpha \frac{\partial C_{kl}^i}{\partial u^\alpha} + F_k^\alpha \frac{\partial C_{lj}^i}{\partial u^\alpha} + F_l^\alpha \frac{\partial C_{jk}^i}{\partial u^\alpha} = \left(C_{mj}^i C_{kl}^m + C_{mk}^i C_{lj}^m + C_{ml}^i C_{jk}^m \right)$$

and

$$(3.4) \qquad F_i^\beta \frac{\partial F_j^\alpha}{\partial u^\beta} - F_j^\beta \frac{\partial F_i^\alpha}{\partial u^\beta} = C_{ij}^l\, F_l^\alpha.$$

Cartan [5] proved a converse:

Theorem 2 (Cartan's Generalized Third Fundamental Theorem)**.** *Let $C_{jk}^i = -C_{kj}^i$ and F_i^α be real-analytic functions on $\mathbb{R}^s$ that satisfy (3.3) and (3.4). Then, for any $u_0 \in \mathbb{R}^s$, there exists an augmented coframing (a, ω) on $\mathbb{R}^n$ that satisfies (3.2) and has $a(0) = u_0$. (Moreover, any two such augmented coframings agree on a neighborhood of $0 \in \mathbb{R}^n$ up to a diffeomorphism of $\mathbb{R}^n$ that fixes $0 \in \mathbb{R}^n$.)*

Proof. Let $M = \mathbb{R}^n \times \mathrm{GL}(n, \mathbb{R}) \times \mathbb{R}^s$, and let $x : M \to \mathbb{R}^n$, $p : M \to \mathrm{GL}(n, \mathbb{R})$, and $u : M \to \mathbb{R}^s$ be the projections. Consider the ideal $\mathcal{I}$ generated on M by the n 2-forms

$$\Upsilon^i = \mathrm{d}(p_j^i\, \mathrm{d}x^j) + \tfrac{1}{2} C_{jk}^i(u)(p_l^j\, \mathrm{d}x^l) \wedge (p_m^k\, \mathrm{d}x^m)$$

and the s 1-forms

$$\theta^\alpha = \mathrm{d}u^\alpha - F_i^\alpha(u)\, (p_j^i\, \mathrm{d}x^j).$$

Note that one can write

$$\Upsilon^i = \pi_j^i \wedge \mathrm{d}x^j$$

for some 1-forms $\pi_j^i = \mathrm{d}p_j^i + P_{jk}^i\, \mathrm{d}x^k$ for some functions P_{jk}^i on M and that the forms π_j^i, $\mathrm{d}x^k$, and θ^α define a coframing on M, i.e., they are linearly independent everywhere and span the cotangent space everywhere.

Now, the hypothesis that $\mathrm{d}^2 = 0$ be a formal consequence of the structure equations (i.e., the equations (3.3) and (3.4)) is easily seen to be equivalent to the equations

$$\mathrm{d}\Upsilon^i = \tfrac{1}{2} \frac{\partial C_{jk}^i}{\partial u^\alpha} \theta^\alpha \wedge (p_l^j\, \mathrm{d}x^l) \wedge (p_m^k\, \mathrm{d}x^m) + C_{jk}^i\, \Upsilon^j \wedge (p_m^k\, \mathrm{d}x^m)$$

and

$$\mathrm{d}\theta^\alpha = \frac{\partial F_i^\alpha}{\partial u^\beta} \theta^\beta \wedge (p_j^i\, \mathrm{d}x^j) + F_i^\alpha\, \Upsilon^i.$$

Thus, these hypotheses imply that $\mathcal{I}$ is generated *algebraically* by the Υ^i and the θ^α. This makes it easy to choose an integral element and compute the Cartan characters:

Fix a point $z \in M$ and let $E \subset T_z M$ be the n-dimensional integral element defined by $\pi_j^i = \theta^\alpha = 0$. Let F be the flag in E defined so that E_i is also annihilated by the $\mathrm{d}x^j$ for $j > i$. Then one finds that $H(E_i)$ is defined by $\theta^\alpha = \pi_k^j = 0$ where $k \leq i$, and hence that $c(E_i) = s + ni$ for $0 \leq i \leq n-1$. In particular, it follows that $\mathcal{V}_n(\mathcal{I})$ must be contained in a submanifold $\mathrm{Gr}_n(TM)$ of codimension at least $C = ns + \frac{1}{2}n^2(n-1)$.

Meanwhile, any n-plane $\tilde{E}$ on which the $\mathrm{d}x^i$ are linearly independent is specified by knowing the $ns + n^3$ numbers $s_i^\alpha(\tilde{E})$ and $s_{jk}^i(\tilde{E})$ such that $\tilde{E}$ satisfies

$$\pi_j^i - s_{jk}^i(\tilde{E})\, \mathrm{d}x^k = \theta^\alpha - s_k^\alpha(\tilde{E})\, \mathrm{d}x^k = 0.$$

The condition that such an $\tilde{E}$ be an integral element of $\mathcal{I}$ is then that $s_k^\alpha(\tilde{E}) = s_{jk}^i(\tilde{E}) - s_{kj}^i(\tilde{E}) = 0$, which is $ns + \frac{1}{2}n^2(n-1) = C$ equations on $\tilde{E}$. Thus, E is ordinary, and F is a regular flag.

Now, since the functions C_{jk}^i and F_i^α are assumed to be real-analytic, the Cartan-Kähler Theorem applies and one concludes that there is an integral manifold of $\mathcal{I}$ tangent to E. This integral manifold is described by having the p_j^i and the u^α be certain functions of the $x^1, \ldots, x^n$, say, $p_j^i = f_j^i(x)$ and $u^\alpha = a^\alpha(x)$. These then give the desired $(a^\alpha, \omega^i) = \left(a^\alpha(x), f_j^i(x)\, \mathrm{d}x^j\right)$. $\qquad\square$

Remark 6 (Cartan's original theorem). The result just proved[5] is only a very special case of the theorem that Cartan proves in the first of his 'infinite groups' papers [5]. However, this version suffices for applications that I have in mind in these notes, and it is much easier to state than Cartan's full theorem.

Remark 7 (Uniqueness). The general ordinary integral of $\mathcal{I}$ depends on n arbitrary functions of n variables (since the last nonzero character is $s_n = n$), but this is to be expected because, given any integral as a (local) coframing on $\mathbb{R}^n$, one can get others by simply pulling back by an arbitrary diffeomorphism of $\mathbb{R}^n$.[6] To get uniqueness up to local diffeomorphism for data (a, ω) in which a takes on a specific value $a_0 \in \mathbb{R}^s$, one shows that two such solutions are locally equivalent by an application of Cartan's technique of the graph.

Note, by the way, that when $s = 0$ (i.e., there are no functions a^α), this result becomes Lie's Third Theorem giving the existence of a local Lie group for any given Lie algebra.

Remark 8 (Smoothness and Globalization). While this treatment assumes real-analyticity, so that the Cartan-Kähler Theorem can be applied, it is now known that the theorem is true in the smooth category as well. The proof in the smooth case is not difficult, but requires a little more insight than this simple application of Cartan-Kähler.

The reader will probably also have noticed that nothing is really used about the domain of the functions C_{jk}^i and F_i^α other than that it is a smooth manifold of some dimension s. This observation spurred the development of a 'globalized' version of Cartan's Theorem, which becomes the subject of *Lie algebroids*, in which $\mathbb{R}^s$ is

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[5]The proof in the text is Cartan's; I have merely simplified his proof as possible in this special case.

[6]Alternatively, one should think of the ordinary integral manifolds of $\mathcal{I}$ as giving a (local) augmented coframing (a, ω) satisfying the structure equations (3.2) *plus* a local coordinate system $x = (x^i)$ on the domain of (a, ω).

replaced by a smooth manifold A. For details on these developments, as well as the smooth theory, the reader should consult treatises devoted to Lie algebroids, but I will sketch the translation here to aid in comparison with a somewhat generalized construction associated to a variant of Cartan's Theorem that I will describe in the next subsection.

Recall that a *Lie algebroid* is a manifold A endowed with a vector bundle $Y \to A$ of rank n whose space of sections $\Gamma(Y)$ carries a Lie algebra structure

$$\{,\} \colon \Gamma(Y) \times \Gamma(Y) \to \Gamma(Y)$$

together with a bundle map $\alpha : Y \to TA$ that induces a homomorphism of Lie algebras on the spaces of sections[7] and that satisfies the Leibnitz compatibility condition

(3.5) $$\{U, fV\} = \alpha(U)f\ V + f\{U, V\}$$

for all $U, V \in \Gamma(Y)$ and $f \in C^\infty(A)$.

A *realization* of $(A, Y, \{,\}, \alpha)$ is a triple (M, a, ω), where M is an n-manifold, $a : M \to A$ is a (smooth) mapping, and $\omega : TM \to a^*Y$ is a vector bundle isomorphism, such that $\alpha \circ \omega = \mathrm{d}a : TM \to TA$ and such that ω induces an isomorphism of Lie algebras on the space of sections of TM and a^*Y.

To see the translation from Cartan's language to that of Lie algebroids, start with the data of functions $C^i_{jk} = -C^i_{kj}$ and F^α_i on $A = \mathbb{R}^s$. Set $Y = A \times \mathbb{R}^n$ with a basis for sections U_i and set

$$\{U_i, U_j\} = C^k_{ij}\, U_k$$

and define $\alpha : Y \to TA = T\mathbb{R}^s$ by

$$\alpha(U_i) = F^\alpha_i\, \frac{\partial}{\partial u^\alpha}.$$

Then (3.3) and (3.4) are precisely the equations necessary and sufficient in order that (3.5) hold, that $\{,\}$ define a Lie bracket on the space of sections of Y, and that $\alpha : Y \to TA$ induce a homomorphism of Lie algebras.

Moreover, an augmented coframing (a^α, ω^i) on a manifold M^n satisfies Cartan's structure equations if and only if, when one sets

$$\omega = U_i\, \omega^i\,,$$

and defines $a : M \to \mathbb{R}^s$ to be $a = (a^\alpha)$, the data (M, a, ω) is a realization in the above sense.

This approach to globalizing Cartan's theorem has been very fruitful, and the reader is encouraged to consult the literature on Lie algebroids for more on this development.

However, it should be borne in mind that Cartan's original formulation in terms of what I am calling 'augmented coframings' turns out already to be very well suited for applications to differential geometry, as I hope to show in the discussion of examples below.

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[7]Here, $\Gamma(TA)$, the set of vector fields on A, is given its standard Lie algebra structure via the Lie bracket.

3.2. **Variants of Cartan's Theorem.** Theorem 2 is one of a number of existence results that are all proved more or less the same way, at least in the real-analytic category. In this subsection, I will give two such variants, and, in the following sections in the notes, I will illustrate their use in a range of differential geometry problems.

Throughout this first variant, the index ranges $1 \le i, j, k \le n$, $1 \le \alpha \le s$, and $1 \le \rho, \sigma \le r$ will be assumed.

Suppose that $C^i_{jk}(u) = -C^i_{kj}(u)$ are given functions on $\mathbb{R}^s$ while $F^\alpha_i(u, v)$ are given functions on $\mathbb{R}^{s+r}$, and suppose that one wants to know whether or not there exist linearly independent 1-forms ω^i on an n-manifold M, a function $a = (a^\alpha) : M \to \mathbb{R}^s$, and a function $b = (b^\rho) : M \to \mathbb{R}^r$ that satisfy these *Cartan structure equations*

$$(3.6) \qquad \mathrm{d}\omega^i = -\tfrac{1}{2} C^i_{jk}(a)\, \omega^j \wedge \omega^k \qquad \text{and} \qquad \mathrm{d}a^\alpha = F^\alpha_i(a, b)\, \omega^i.$$

Such a triple (a, b, ω) on M will be said to be an *augmented coframing* satisfying (3.6).

Note that this is a diffeomorphism invariant notion, since, if $f : N \to M$ is a diffeomorphism, then $\left(f^*a, f^*b, f^*\omega\right)$ will be an augmented coframing on N that satisfies (3.6). In many geometric problems (see some examples in the next section), one is interested in understanding the 'general' augmented coframing satisfying (3.6) and one regards two such augmented coframings that differ by a diffeomorphism as equivalent.

One should think of the b^ρ as 'unconstrained' derivatives of the functions a^α. Thus, this version of Cartan's structure equations covers situations in the more typical case in which one does not have formulae for all of the derivatives of the geometric quantities that appear in the problem. Informally, one speaks of the functions b^ρ as 'free derivatives'.

To understand necessary and sufficient conditions for such augmented coframings to exist, one again wants to consider the consequences of the identity $\mathrm{d}^2 = 0$, but now, because of the free derivatives appearing in the structure equations, one cannot simply expand this fundamental identity formally and arrive at complete necessary and sufficient conditions on the functions C and F.

Now, the equations $\mathrm{d}(\mathrm{d}\omega^i) = \mathrm{d}(C^i_{jk}(a)\, \omega^j \wedge \omega^k) = 0$ do make good sense, so one should require that C and F at least satisfy

$$(3.7) \qquad F^\alpha_j \frac{\partial C^i_{kl}}{\partial u^\alpha} + F^\alpha_k \frac{\partial C^i_{lj}}{\partial u^\alpha} + F^\alpha_l \frac{\partial C^i_{jk}}{\partial u^\alpha} = \left(C^i_{mj} C^m_{kl} + C^i_{mk} C^m_{lj} + C^i_{ml} C^m_{jk} \right).$$

(Because the F^α_i contain the variables v^ρ while the right hand side of (3.7) does not, this equation places constraints on how the v^ρ can appear in the F^α_j.)

Meanwhile, expanding $\mathrm{d}(\mathrm{d}a^\alpha) = \mathrm{d}\left(F^\alpha_i(a, b)\, \omega^i\right) = 0$ yields

$$0 = \frac{\partial F^\alpha_i}{\partial v^\rho}(a, b)\mathrm{d}b^\rho \wedge \omega^i + \frac{1}{2} \left(F^\beta_i(a, b)\frac{\partial F^\alpha_j}{\partial u^\beta}(a, b) - F^\beta_j(a, b)\frac{\partial F^\alpha_i}{\partial u^\beta}(a, b) - C^l_{ij}(a)\, F^\alpha_l(a, b) \right) \omega^i \wedge \omega^j,$$

and the simplest way for these equations to be satisfiable by some expression of the $\mathrm{d}b^\rho$ in terms of the ω^i would be for there to exist functions G^ρ_j on $\mathbb{R}^{s+r}$ such that

$$(3.8) \qquad F^\beta_i \frac{\partial F^\alpha_j}{\partial u^\beta} - F^\beta_j \frac{\partial F^\alpha_i}{\partial u^\beta} - C^l_{ij}\, F^\alpha_l = \frac{\partial F^\alpha_i}{\partial v^\rho} G^\rho_j - \frac{\partial F^\alpha_j}{\partial v^\rho} G^\rho_i \,,$$

for then the above equations can be written in the form

$$0 = \frac{\partial F_i^\alpha}{\partial v^\rho}(a,b) \left(\mathrm{d}b^\rho - G_j^\rho(a,b)\omega^j\right) \wedge \omega^i.$$

The conditions (3.7) and (3.8) will at least ensure that there are no obvious incompatibilities derivable by taking the exterior derivatives of the structure equations. However, they aren't enough to guarantee that there won't be higher order incompatibilities. To rule this out, it will be necessary to impose conditions on how the 'free derivatives' v^ρ appear in the functions F_i^α. Let $u_1, \ldots, u_s$ be a basis of $\mathbb{R}^s$, and let $x^1, \ldots x^n$ be a basis of the dual of $\mathbb{R}^n$. Let $A(u,v) \subset \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^s)$ denote the subspace (i.e., tableau) spanned by the r elements

$$(3.9) \qquad \frac{\partial F_i^\alpha}{\partial v^\rho}(u,v)\, u_\alpha \otimes x^i, \qquad 1 \le \rho \le r.$$

This $A(u,v)$ is known as the 'tableau of free derivatives' of the structure equations at the point $(u,v) \in \mathbb{R}^{s+r}$.

Here, then, is a useful variant[8] of Theorem 2.

Theorem 3. *Suppose that real analytic functions $C_{jk}^i = -C_{kj}^i$ on $\mathbb{R}^s$ and F_i^α on $\mathbb{R}^{s+r}$ are given satisfying (3.7) and that there exist real analytic functions G_i^ρ on $\mathbb{R}^{s+r}$ that satisfy (3.8). Finally, suppose that the tableaux $A(u,v)$ defined by (3.9) have dimension r and are involutive, with Cartan characters s_i $(1 \le i \le n)$ for all $(u,v) \in \mathbb{R}^{s+r}$. Then, for any $(u_0, v_0) \in \mathbb{R}^{s+r}$ there exists an augmented coframing (a, b, ω) on an open neighborhood V of 0 in $\mathbb{R}^n$ that satisfies (3.6) and has $\big(a(0), b(0)\big) = (u_0, v_0)$.*

Proof. Let $M = \mathbb{R}^n \times \mathrm{GL}(n, \mathbb{R}) \times \mathbb{R}^s \times \mathbb{R}^r$, and let $x : M \to \mathbb{R}^n$, $p : M \to \mathrm{GL}(n, \mathbb{R})$, $u : M \to \mathbb{R}^s$, and $v : M \to \mathbb{R}^r$ be the projections. Consider the ideal $\mathcal{I}$ generated on M by the n 2-forms

$$\Upsilon^i = \mathrm{d}(p_j^i\, \mathrm{d}x^j) + \tfrac{1}{2}C_{jk}^i(u)(p_l^j\, \mathrm{d}x^l) \wedge (p_m^k\, \mathrm{d}x^m)$$

and the s 1-forms

$$\theta^\alpha = \mathrm{d}u^\alpha - F_i^\alpha(u,v)\,(p_j^i\, \mathrm{d}x^j).$$

Note that, as in the proof of Theorem 2, one can write

$$\Upsilon^i = \pi_j^i \wedge \mathrm{d}x^j$$

for some 1-forms $\pi_j^i = \mathrm{d}p_j^i + P_{jk}^i\, \mathrm{d}x^k$ for some functions P_{jk}^i on M and that the forms π_j^i, $\mathrm{d}x^k$, θ^α, together with $\beta^\rho = \mathrm{d}b^\rho - G_i^\rho(p_j^i\, \mathrm{d}x^j)$ define a coframing on M, i.e., they are linearly independent everywhere and span the cotangent space everywhere.

Now, the hypotheses of the theorem imply that

$$\mathrm{d}\Upsilon^i = \tfrac{1}{2}\frac{\partial C_{jk}^i(u)}{\partial a^\alpha}\, \theta^\alpha \wedge (p_l^j\, \mathrm{d}x^l) \wedge (p_m^k\, \mathrm{d}x^m) + C_{jk}^i(u)\, \Upsilon^j \wedge (p_m^k\, \mathrm{d}x^m)$$

while

$$\mathrm{d}\theta^\alpha = \frac{\partial F_i^\alpha}{\partial b^\rho}\beta^\rho \wedge (p_j^i\, \mathrm{d}x^j) + \frac{\partial F_i^\alpha(u)}{\partial a^\beta}\, \theta^\beta \wedge (p_j^i\, \mathrm{d}x^j) + F_i^\alpha\, \Upsilon^i.$$

Thus, $\mathcal{I}$ is generated *algebraically* by the Υ^i, the θ^α, and the 2-forms

$$\Theta^\alpha = \frac{\partial F_i^\alpha}{\partial b^\rho}\beta^\rho \wedge (p_j^i\, \mathrm{d}x^j).$$

This makes it easy to choose an integral element and compute the Cartan characters:

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[8]Note that the proof is very closely patterned on Cartan's proof of Theorem 2.

Fix a point $z = (0, I_n, u_0, v_0) \in M$, and let $E \subset T_z M$ be the n-dimensional integral element defined by $\pi_j^i = \theta^\alpha = \beta^\rho = 0$.

Choose a regular flag for the tableau $A(u_0, v_0)$ (which, by hypothesis, exists). By rotating the x^i if necessary, one can assume that the flag F in E defined so that E_i is also annihilated by the $\mathrm{d}x^j$ for $j > i$ is such a regular flag. Then one finds that $H(E_i)$ is defined by

$$\theta^\alpha = \pi_k^j = \frac{\partial F_k^\alpha}{\partial b^\rho} \beta^\rho = 0$$

where $k \le i$, so $c(E_i) = s + ni + \dim A(u_0, v_0)_i = s + ni + s_1 + \cdots + s_i$ for $0 \le i \le n-1$.

Meanwhile, any n-plane $E' \in \mathrm{Gr}_n(TM)$ on which the $\mathrm{d}x^i$ are independent will be defined by equations of the form

$$\theta^\alpha - q_i^\alpha(E')\,\mathrm{d}x^i = \pi_k^l - q_{ki}^l(E')\,\mathrm{d}x^i = \beta^\rho - q_i^\rho(E')\,(p_j^i\,\mathrm{d}x^j) = 0$$

for some numbers $q_i^\alpha(E'), q_{ki}^l(E'), q_i^\rho(E')$. The conditions that E' be an integral element of $\mathcal{I}$ then imply that

$$q_i^\alpha(E') = q_{ki}^l(E') - q_{ik}^l(E') = \frac{\partial F_i^\alpha}{\partial b^\rho}(u, v) q_j^\rho(E') - \frac{\partial F_j^\alpha}{\partial b^\rho}(u, v) q_i^\rho(E') = 0,$$

and, by the hypothesis that $F = (E_i)$ is a regular flag for the tableaux $A(u_0, v_0)$ (and hence is also regular for $A(u, v)$ for (u, v) near (u_0, v_0)), it follows that this is

$$c(E_0) + c(E_1) + \cdots + c(E_{n-1}) = ns + \tfrac{1}{2}n^2(n-1) + (n-1)s_1 + (n-2)s_2 + \cdots + s_{n-1}$$

equations on the quantities $q_i^\alpha(E'), q_{ki}^l(E'), q_i^\rho(E')$. Thus, Cartan's Bound is saturated, and the flag F is regular for E.[9]

The ideal $\mathcal{I}$ is real analytic, so the Cartan-Kähler Theorem applies, and one concludes that there is an integral manifold of $\mathcal{I}$ tangent to E. This integral manifold is described by having the p_j^i, the u^α, and the v^ρ be functions of the $x^1, \ldots, x^n$, say, $p_j^i = f_j^i(x)$ and $u^\alpha = a^\alpha(x)$ and $v^\rho = b^\rho(x)$. These then give the desired augmented coframing $(a^\alpha, b^\rho, \omega^i) = \left(a^\alpha(x),\ b^\rho(x),\ f_j^i(x)\,\mathrm{d}x^j\right)$. $\qquad\square$

Remark 9 (Generality). Theorem 3 as stated only gives existence for specified (u_0, v_0), but, as will be seen, the (local) augmented coframings that satisfy the structure equations depend (modulo diffeomorphism) on s constants, s_1 functions of 1 variable, s_2 functions of 2 variables, etc., but to make precise sense of this, I will need to discuss *prolongation*, which comes in the next section.

Remark 10 (Globalization). Just as in the case of Theorem 2, which has a modern formulation in terms of Lie algebroids, there is a 'global' version of Theorem 3.[10]

The appropriate global data structure, $(A, B, \pi, Y, \{,\}, \beta)$, starts with two manifolds, A of dimension s and B of dimension $r+s$, and a submersion $\pi : B \to A$. For notational convenience, let $K = \ker \pi' \subset TB$, and let $Q = TB/K$ be the quotient bundle over B. For a vector field X on B, let X_K (i.e., 'X modulo K') denote the corresponding section of Q.

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[9]Essentially, the involutive tableau $A(u_0, v_0)$ is being combined with a tableau already shown to be involutive in the proof of Theorem 2, one for which *every* flag is regular. Perhaps, I should also remind the reader that the s_i are the characters of the tableaux $A(u, v)$ and *not* of the ideal $\mathcal{I}$ constructed above. In fact, one has $s_0(\mathcal{I}) = s$ and $s_i(\mathcal{I}) = s_i + n$ for $1 \le i \le n$.

[10]I will not actually need this formulation in these notes, but since there were questions about this during the lectures, I will briefly describe it here.

Next, the data structure includes a vector bundle $Y \to A$ of rank n, and a Lie algebra structure $\{,\}$ on the space $C^\infty(Y)$ of sections of Y over A. For $U \in C^\infty(Y)$, let $U^\pi \in C^\infty(\pi^* Y)$ denote the pullback section of the pullback bundle over B, i.e., $U^\pi(b) = U\big(\pi(b)\big)$.

Finally, the data includes a bundle map $\beta : \pi^* Y \to TB$, that satisfies

$$\tag{3.10} \beta\big(\{U, V\}^\pi\big)_K = \big[\beta(U^\pi), \beta(V^\pi)\big]_K$$

and the requirement that there exist an anti-symmetric, bilinear product[11] $\{\{,\}\}$ on $C^\infty(\pi^* Y)$ that satisfies the compatibility condition

$$\tag{3.11} \{\{U^\pi, f\, V^\pi\}\} = \big(\beta(U^\pi)f\big)\, V^\pi + f\, \{U, V\}^\pi$$

for $U, V \in C^\infty(Y)$ and $f \in C^\infty(B)$.

A *realization* of the data structure $\big(A, B, \pi, Y, \{,\}, \beta\big)$ is a triple (M, b, ω), where M is an n-manifold, $b : M \to B$ is a smooth mapping, and $\omega : TM \to (\pi \circ b)^* Y$ is an isomorphism of bundles that induces an isomorphism of Lie algebras on the appropriate spaces of sections and that satisfies $\mathrm{d}(\pi \circ b) = \pi' \circ \beta \circ \omega$.

(Note that if $\pi : B \to A$ is a diffeomorphim (e.g., $r = 0$), then the data $\big(A, Y, \{,\}, \beta\big)$ defines a Lie algebroid, and the notion of a realization is the standard one.)

Now, there is a map $\tau : K \to Q \otimes (\pi^* Y)^*$ of B-bundles, uniquely determined by the condition that it satisfy

$$\tau(X)(U^\pi) = [X, \beta(U^\pi)]_K$$

for any $X \in \Gamma(K)$ and $U \in C^\infty(Y)$. One says that the data $\big(A, B, \pi, Y, \{,\}, \beta\big)$ is *nondegenerate* if τ injective, and, further, that it is (*uniformly*) *involutive* if $\tau(K)_b \subset Q_b \otimes (\pi^* Y)_b^*$ is an involutive tableau for all $b \in B$ (and the Cartan characters $s_i\big(\tau(K)_b\big)$ are constant, independent of $b \in B$).

Then Theorem 3 asserts the local existence of realizations (M, b, ω) of uniformly involutive, nondegenerate real analytic data structures $\big(A, B, \pi, Y, \{,\}, \beta\big)$ with $b : M \to B$ taking any specified value $b_0 \in B$.

To see the translation from the notation of Theorem 3 to this 'global' formulation, let $A = \mathbb{R}^s$ (with coordinates u^α), let $B = \mathbb{R}^{s+r}$ (with coordinates u^α and v^ρ), let $\pi : \mathbb{R}^{s+r} \to \mathbb{R}^s$ be the projection on the first s coordinates, let $Y = \mathbb{R}^s \times \mathbb{R}^n$ (with the standard basis of sections U_i), let

$$\{U_i, U_j\} = C_{ij}^k(u)\, U_k\,,$$

and let

$$\beta(U_i^\pi) = F_i^\alpha(u, v) \frac{\partial}{\partial u^\alpha} + G_i^\rho(u, v) \frac{\partial}{\partial v^\rho}.$$

The reader can now verify that (3.7) and (3.8) are the necessary and sufficient conditions that $\{,\}$ define a Lie bracket on $C^\infty(Y)$, that (3.10) hold, and that there exists an extension $\{\{,\}\}$ of $\{,\}$ to sections of $C^\infty(\pi^* Y)$ that satisfies (3.11).

(I should point out that this 'global' formulation is not perfect, because, ideally, one should only have to specify the functions G_i^ρ up to a section of the prolongation of the tableau bundle, i.e., one should regard two such structures $\big(A, B, \pi, Y, \{,\}, \beta\big)$

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[11]N.B.: It is easy to see that there is at most one such product $\{\{,\}\}$ satisfying (3.11). In general, this 'extended' product is *not* a Lie algebra structure on $C^\infty(\pi^* Y)$.

and $\big(A,B,\pi,Y,\{,\},\tilde\beta\big)$ as the same if the difference $\delta\beta = \tilde\beta - \beta$, which is a section of $TB \otimes (\pi^*Y)^*$, is actually a section of the kernel $K^{(1)}$ of the composition

$$K \otimes (\pi^*Y)^* \stackrel{\tau \otimes \mathrm{id}}{\longrightarrow} Q \otimes (\pi^*Y)^* \otimes (\pi^*Y)^* \to Q \otimes \Lambda^2\big((\pi^*Y)^*\big).$$

Thus, one should probably formulate the data structure with the notion of non-degeneracy built into the axioms and with β taking values in the quotient bundle $(TB \otimes (\pi^*Y)^*)/K^{(1)}$ instead of in $TB \otimes (\pi^*Y)^*$. However, this is turns out to be awkward, as checking that the axioms even make sense becomes cumbersome.)

While this 'global' formulation may be more satisfying than the 'coordinate' formulation in Theorem 3, one should bear in mind that there is little (and, more often than not, no) hope of proving the global realization theorems that one has in the more familiar case of Lie algebroids. For the general such structure, there is no obvious notion of completeness of a realization and there is also no obvious way to 'classify' even the germs of realizations up to diffeomorphism. (However, there is a way to test two such germs for diffeomorphism equivalence, at least in the real-analytic category. I will say more about this in Remark 11.)

Remark 11 (Local equivalence of realizations). The reader may be wondering how one distinguishes two realizations of the data in Theorem 3 up to diffeomorphism. After all, as Cartan proved, given two augmented coframings (M,a,ω) and $(\bar M,\bar a,\bar\omega)$ satisfying (3.2) and points $x \in M$ and $\bar x \in \bar M$ such that $a(x) = \bar a(\bar x)$, there will exist an x-neighborhood $U \subset M$, an $\bar x$-neighborhood $\bar U \subset \bar M$, and a diffeomorphism $f : \bar U \to U$ such that $(\bar a,\bar\omega) = (f^*a, f^*\omega)$ and $f(\bar x) = x$.

In contrast, for augmented coframings (M,a,b,ω) and $(\bar M,\bar a,\bar b,\bar\omega)$ satisfying (3.6), having points $x \in M$ and $\bar x \in \bar M$ with $\big(a(x),b(x)\big) = \big(\bar a(\bar x),\bar b(\bar x)\big)$ is not sufficient to imply that there is a diffeomorphism $f : \bar U \to U$ for some x-neighborhood U and $\bar x$-neighborhood $\bar U$ such that $(\bar a,\bar b,\bar\omega) = (f^*a, f^*b, f^*\omega)$.

A sufficient condition (due, of course, to Cartan [7]) for local diffeomorphism equivalence does exist in this more general case but it is more subtle.

An augmented coframing (a,b,ω) on M^n satisfying (3.6), is *regular of rank p at $x \in M$* if there is an x-neighborhood $U \subset M$, a smooth submersion $h : U \to \mathbb{R}^p$, and a smooth map $(A,B) : h(U) \to \mathbb{R}^{s+r}$ such that $A : h(U) \to \mathbb{R}^s$ is a smooth embedding and such that $(a,b) = (A{\circ}h, B{\circ}h)$ holds on U. Note, in particular, that this implies that the image $(a,b)(U) \subset \mathbb{R}^{s+r}$ is a smoothly embedded p-dimensional submanifold that is a graph over its projection $a(U) \subset \mathbb{R}^s$ (also a smoothly embedded p-dimensional submanifold). Equivalently, (a,b,ω) is regular of rank p at $x \in M$ if some p of the functions a^α have independent differentials at x and, moreover, on some x-neighborhood $U \subset M$, all of the other a^α *and* all of the b^ρ can be expressed as smooth functions of those p independent functions. For an augmented coframing (a,b,ω) satisfying (3.6), being regular of rank p at a point $x \in M$ is a diffeomorphism-invariant condition.

Cartan showed that, *if* (M,a,b,ω) and $(\bar M,\bar a,\bar b,\bar\omega)$ satisfy (3.6), are regular of rank p at points $x \in M$ and $\bar x \in \bar M$ with $\big(a(x),b(x)\big) = \big(\bar a(\bar x),\bar b(\bar x)\big)$, and there are an x-neighborhood $U \subset M$ and $\bar x$-neighborhood $\bar U \subset \bar M$ such that $(a,b)(U)$ and $(\bar a,\bar b)(\bar U)$ are the same p-dimensional submanifold of $\mathbb{R}^{r+s}$, *then*, after possibly shrinking U and $\bar U$, there exists a diffeomorphism $f : \bar U \to U$ such that $(\bar a,\bar b,\bar\omega) = (f^*a, f^*b, f^*\omega)$ and $f(\bar x) = x$.

The reader should have no trouble rephrasing Cartan's sufficient condition in a form suitable for the 'global data structure' version described in Remark 10. (The

reader may feel that the hypotheses of Cartan's equivalence theorem are absurdly strong, but, without knowing more about a specific set of structure equations (3.6), it is not possible to weaken these hypotheses in any significant way and still get the conclusion of local equivalence, as examples show.)

I conclude this subsection with another useful variant of Cartan's Theorem. Let V be a vector space of dimension n. For each V-valued coframing $\omega : TM \to V$ on an n-manifold M, there will be a unique function $C : M \to V \otimes \Lambda^2(V^*)$, the *structure function* of ω, such that

$$(3.12) \qquad\qquad d\omega = -\tfrac{1}{2}C(\omega \wedge \omega).$$

Given a basis v_i of V with dual basis v^i, one has $\omega = v_i\omega^i$ and $C = \tfrac{1}{2}C^i_{jk}v_i \otimes v^j \wedge v^k$, and (3.12) takes the familiar form $d\omega^i = -\tfrac{1}{2}C^i_{jk}\,\omega^j \wedge \omega^k$.

Now, let $A \subset V \otimes \Lambda^2(V^*)$ be a submanifold.[12] A V-valued coframing $\omega : TM \to V$ will be said to be of *type A* if its structure function $C : M \to V \otimes \Lambda^2(V^*)$ takes values in A. The goal is to determine the generality of the space of (local) V-valued coframings ω of type A when two such that differ by a diffeomorphism of M are regarded as equivalent.

For example, if A consists of a single point $a_0 = \tfrac{1}{2}c^i_{jk}\,v_i \otimes v^j \wedge v^k$, then Lie's Theorem asserts that a necessary and sufficient condition that such a coframing exist is that $J(a_0) = 0$, where $J : V \otimes \Lambda^2(V^*) \to V \otimes \Lambda^3(V^*)$ is the quadratic mapping (sometimes called the *Jacobi mapping*) that one gets by squaring, contracting, and skewsymmetrizing:

$$V \otimes \Lambda^2(V^*) \to \big(V \otimes \Lambda^2(V^*)\big) \otimes \big(V \otimes \Lambda^2(V^*)\big) \to V \otimes V^* \otimes \Lambda^2(V^*) \to V \otimes \Lambda^3(V^*).$$

Given a basis v_i of V with dual basis v^i, the formula for J is

$$J\big(\tfrac{1}{2}c^i_{jk}\,v_i \otimes v^j \wedge v^k\big) = \tfrac{1}{6}(c^i_{jm}c^m_{kl} + c^i_{km}c^m_{lj} + c^i_{lm}c^m_{jk})\,v_i \otimes v^j \wedge v^k \wedge v^l.$$

Of course, in this case, *all* V-valued coframings of type $A = \{a_0\}$ are locally equivalent up to diffeomorphism.

This motivates the following definitions: A submanifold $A \subset V \otimes \Lambda^2(V^*)$ is said to be a *Jacobi manifold* if

$$(3.13) \qquad\qquad J(a) \in \sigma(T_aA \otimes V^*)$$

for all $a \in A$, where $\sigma : V \otimes \Lambda^2(V^*) \otimes V^* \to V \otimes \Lambda^3(V^*)$ is the skewsymmetrization mapping defined by exterior multiplication. The condition (3.13) is an obvious necessary condition in order for there to exist a V-valued coframing $\omega : TM \to V$ whose structure function takes values in A and assumes the value $a \in A$. It is not, in general, sufficient.

A Jacobi manifold A is *involutive* if each of its tangent spaces $T_aA \subset V \otimes \Lambda^2(V^*)$ is involutive, with characters $s_i(T_aA) = s_i$ (independent of $a \in A$).

I can now state a useful existence result that will be applied in some examples in the final section.

Theorem 4. *Let V be a vector space, and let $A \subset V \otimes \Lambda^2(V^*)$ be a real-analytic, involutive Jacobi manifold. Then, for any $a_0 \in A$, there exists a V-valued coframing ω of type A on a neighborhood U of $0 \in V$ such that its structure function C satisfies $C(0) = a_0$.*

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[12]In most applications, A will be an affine subspace of $V \otimes \Lambda^2(V^*)$, but the extra generality of allowing A to be a submanifold is frequently useful.

Proof. The proof follows the by-now familiar pattern laid down by Cartan.[13]

The result is local, so one can suppose that A has dimension s and is parametrized by a real-analytic embedding $f : \mathbb{R}^s \to A \subset V \otimes \Lambda^2(V^*)$ with $f(0) = a_0 \in A$. Write

$$f = \tfrac{1}{2} f^i_{jk}\, v_i \otimes v^j \wedge v^k$$

where the $f^i_{jk} = -f^i_{kj}$ are real analytic functions on $\mathbb{R}^s$. By hypothesis, for each $u_0 = (u^\alpha_0) \in \mathbb{R}^s$, the tableau $T_{f(u_0)}A \subset V \otimes \Lambda^2(V^*)$, which is spanned by the s independent elements

$$\frac{\partial f^i_{jk}}{\partial u^\alpha}(u_0)\, v_i \otimes v^j \wedge v^k\,, \qquad 1 \le \alpha \le s,$$

is involutive, with characters s_i for $1 \le i \le n$. By changing the basis of V if necessary, it can even be supposed that the flag such that $V_i \subset V$ is spanned by $v_1, \ldots, v_i$ is a regular flag for $T_{f(0)}A$ (and hence it will be regular for $T_{f(u_0)}A$ for all u_0 in a neighborhood O of $0 \in \mathbb{R}^s$). For the rest of the proof, I use this basis to identify V with $\mathbb{R}^n$.

Also, by the hypothesis that A be a Jacobi manifold, the linear equations for quantities g^α_i given as

$$\frac{\partial f^i_{kl}}{\partial u^\alpha}(u)\, g^\alpha_j + \frac{\partial f^i_{lj}}{\partial u^\alpha}(u)\, g^\alpha_k + \frac{\partial f^i_{jk}}{\partial u^\alpha}(u)\, g^\alpha_l = \left(f^i_{mj}(u) f^m_{kl}(u) + f^i_{mk}(u) f^m_{lj}(u) + f^i_{ml}(u) f^m_{jk}(u) \right).$$

are solvable, and the associated homogeneous linear system for the g^α_i has, for each value of u, a solution space of dimension $s_1 + 2\, s_2 + \cdots + n\, s_n$. Thus, the equations are compatible and have constant rank, so there exist real-analytic functions $g^\alpha_i(u)$ on a neighborhood of $0 \in \mathbb{R}^s$ (which can be supposed to be O) that furnish solutions to the above inhomogeneous system.

Let $M = \mathbb{R}^n \times \mathrm{GL}(n, \mathbb{R}) \times O$, where $O \subset \mathbb{R}^s$ is the neighborhood of $0 \in \mathbb{R}^s$ selected above. Let $x : M \to \mathbb{R}^n$, $p : M \to \mathrm{GL}(n, \mathbb{R})$, and $a : M \to O$ be the respective projections. Set $\eta^i = p^i_j\, \mathrm{d}x^j$ and $\pi^\alpha = \mathrm{d}u^\alpha - g^\alpha_i(u)\, \eta^i$.

Now let $\mathcal{I}$ be the ideal on M generated by the 2-forms

$$\Upsilon^i = \mathrm{d}\eta^i + \tfrac{1}{2} f^i_{jk}(u)\, \eta^j \wedge \eta^k.$$

Note that there exist 1-forms π^i_j such that $\Upsilon^i = \pi^i_j \wedge \eta^j$ and such that the 1-forms π^i_j, η^i, and π^α are linearly independent and hence define a coframing on M.

Now, by the way the functions g^α_i on O were chosen, one has

$$\mathrm{d}\Upsilon^i = \tfrac{1}{2} \frac{\partial f^i_{jk}}{\partial u^\alpha}(u)\, \pi^\alpha \wedge \eta^j \wedge \eta^k + f^i_{jk}(u)\, \Upsilon^j \wedge \eta^k\,,$$

so that $\mathcal{I}$ is generated algebraically by the 2-forms $\Upsilon^i = \pi^i_j \wedge \eta^j$ and the 3-forms

$$\Psi^i = \frac{\partial f^i_{jk}}{\partial u^\alpha}(u)\, \pi^\alpha \wedge \eta^j \wedge \eta^k\,.$$

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[13]However, the reader will note that, in this proof, I apply the Cartan-Kähler Theorem to a differential ideal generated algebraically in degrees 2 and 3, while Cartan's original version of the Cartan-Kähler Theorem assumed that the ideal would be generated algebraically in degrees 1 and 2. Thus, this proof, strictly speaking, would not have been available to Cartan in his early work on exterior differential systems. It's possible to give a proof that only uses the version of the Cartan-Kähler Theorem that was available to Cartan at that time, but that proof would be somewhat more complicated.

Since A is involutive, the integral elements in $\mathcal{V}_n(\mathcal{I})$ defined at each point of M by $\pi^i_j = \pi^\alpha = 0$ are all ordinary. By the Cartan-Kähler Theorem, there is an n-dimensional integral manifold $\mathcal{I}$ tangent to this integral element at the point $(0, I_n, 0) \in M$.

This integral manifold is written as a graph of the form $\big(x, p^i_j(x), u^\alpha(x)\big)$ for x in a neighborhood of $0 \in \mathbb{R}^n$. Now, setting $\omega^i = p^i_j(x)\,\mathrm{d}x^j$, one sees that the structure function of the coframing $\omega = v_i \omega^i$ is

$$C = \tfrac{1}{2} f^i_{jk}\big(u^\alpha(x)\big)\, v_i \otimes v^j \wedge v^k,$$

which takes values in A and, in particular, takes the value $a_0 \in A$ at $x = 0$. $\qquad\square$

Remark 12 (Checking the hypotheses). Note that, in practical terms, checking the condition that $A \subset V \otimes \Lambda^2(V^*)$ be an involutive Jacobi manifold can be reduced to a relatively simple calculation:

A coframing satisfying the structure equations (3.12) will necessarily satisfy

$$0 = \mathrm{d}(\mathrm{d}\omega) = -\tfrac{1}{2}\mathrm{d}C \wedge (\omega \wedge \omega) + \tfrac{1}{2}C\big(C(\omega \wedge \omega) \wedge \omega\big),$$

or, relative to a basis v_i of V with dual basis v^i,

$$0 = -\tfrac{1}{2}\mathrm{d}C^i_{jk} \wedge \omega^j \wedge \omega^k + \tfrac{1}{6}(C^i_{mj}C^m_{kl} + C^i_{mk}C^m_{lj} + C^i_{ml}C^m_{jk})\,\omega^j \wedge \omega^k \wedge \omega^l.$$

Regarding the v^i as linear coordinates on V and regarding the $C^i_{jk} = -C^i_{kj}$ as the components of the embedding of A into $V \otimes \Lambda^2(V^*)$, one can consider the *algebraic* ideal $\mathcal{I}_A$ generated on $M = A \times V$ by the 3-forms

$$\bar{\Psi}^i = \tfrac{1}{2}\mathrm{d}C^i_{jk} \wedge \mathrm{d}v^j \wedge \mathrm{d}v^k - \tfrac{1}{6}(C^i_{mj}C^m_{kl} + C^i_{mk}C^m_{lj} + C^i_{ml}C^m_{jk})\,\mathrm{d}v^j \wedge \mathrm{d}v^k \wedge \mathrm{d}v^l.$$

(N.B.: Just this once, I do *not* want to consider the differential closure of $\mathcal{I}_A$.)

Then $\mathcal{I}_A$ has an integral element E of dimension n based at $(0, a) \in V \times A$ on which the $\mathrm{d}v^i$ are independent if and only if (3.13) is satisfied. Moreover, this integral element is ordinary iff $T_a A$ is an involutive subspace of $V \otimes \Lambda^2(V^*)$.

Remark 13. It will turn out that the s_i for a involutive Jacobi manifold A have a significance for describing the differential invariants of V-valued coframings taking values in A. As will be shown below, in an appropriate sense, the V-valued coframings whose structure functions take values in A depend (modulo diffeomorphism) on s_1 functions of 1 variable, s_2 functions of 2 variables, etc.

4. Ordinary prolongation

It is time to take a closer look at the geometry of $\mathcal{V}^o_n(\mathcal{I})$.

4.1. The tableau of an ordinary element.
Recall that the basepoint projection $\pi : \mathcal{V}^o_n(\mathcal{I}) \to M$ is a smooth submersion, so the fiber over x, which is $\mathcal{V}^o_n(\mathcal{I}) \cap \mathrm{Gr}_n(T_x M)$, is a smooth submanifold of $\mathrm{Gr}_n(T_x M)$. For a given $E \in \mathcal{V}^o_n(\mathcal{I})$, the tangent space to this fiber is an involutive tableau

$$A_E \subset T_E\,\mathrm{Gr}_n(T_x M) \simeq \big(T_x M/E\big) \otimes E^*$$

of dimension $s_1(E) + 2s_2(E) + \cdots + ns_n(E)$, and its Cartan characters are given by

$$s_i(A_E) = s_i(E) + s_{i+1}(E) + \cdots + s_n(E).$$

4.2. **The ordinary prolongation of $\mathcal{I}$.** Set $M^{(1)} = \mathcal{V}_n^o(\mathcal{I})$. Define a subbundle $C \subset T^*M^{(1)}$ by letting

$$C(E) = \pi^*(E^\perp),$$

where $E^\perp \subset T^*_{\pi(E)}M$ is the annihilator of $E \subset T_{\pi(E)}M$. This subbundle of rank $\dim M - n$ is known as the *contact bundle* on $M^{(1)}$.

Let $\mathcal{I}^{(1)} \subset \mathcal{A}^*\big(M^{(1)}\big)$ denote the differential ideal generated by the sections of C. The ideal $\mathcal{I}^{(1)}$ on $M^{(1)}$ is known as the *ordinary prolongation* of $\mathcal{I}$ on M. (Technically, the definition of the prolongation depends on the choice of n, but, in nearly all applications, the choice of n is determined by the problem that $\mathcal{I}$ was designed to study, so I will not make this part of the notation.)

Every ordinary integral manifold $f : N \to M$ has a canonical lift $f^{(1)} : N \to M^{(1)}$, defined by $f^{(1)}(x) = f'(T_xN) \in \mathcal{V}_n^o(\mathcal{I}) = M^{(1)}$. It follows directly from the definition that $f^{(1)} : N \to M^{(1)}$ is an integral manifold of $\mathcal{I}^{(1)}$ and, moreover, any integral manifold $h : N \to M^{(1)}$ that is an integral of $\mathcal{I}^{(1)}$ and has the property that $\pi \circ h : N \to M$ is an immersion is of the form $h = f^{(1)}$, in fact, with $f = \pi \circ h$.

At the integral element level, every $\tilde{E} \in \mathcal{V}_n\big(\mathcal{I}^{(1)}\big)$ with $\tilde{E} \subset T_E M^{(1)}$ such that $\pi' : \tilde{E} \to T_{\pi(E)}M$ is injective actually satisfies $\pi'(\tilde{E}) = E$. Moreover, each such $\tilde{E}$ is ordinary, with Cartan characters

$$s_i(\tilde{E}) = s_i(E) + s_{i+1}(E) + \cdots + s_n(E),$$

and with a flag $\tilde{F} = (\tilde{E}_0, \ldots, \tilde{E}_{n-1})$ of $\tilde{E}$ being regular if and only if the flag $F = (E_0, \ldots, E_{n-1})$ with $E_i = \pi'(\tilde{E}_i)$ is a regular flag of E.

4.3. **The higher prolongations.** In particular, one can repeat the prolongation process, but, now considering $M^{(2)} \subset \mathcal{V}_n^o\big(\mathcal{I}^{(1)}\big)$ to be the open subset consisting of those $\tilde{E}$ that satisfy the 'transversality' condition $\pi'(\tilde{E}) = E$ (and retaining the corresponding condition for all the higher prolongations, etc). This defines a sequence of manifolds $M^{(k)}$ with ideals $\mathcal{I}^{(k)}$, such that $\big(M^{(0)}, \mathcal{I}^{(0)}\big) = (M, \mathcal{I})$ while, for $k \geq 1$, the manifold $M^{(k)}$ is embedded as an open subset of $\mathcal{V}_n^o\big(\mathcal{I}^{(k-1)}\big)$. By induction, one sees that the ideal $\mathcal{I}^{(k)}$ has Cartan characters

$$s_j^{(k)} = s_j + \binom{k}{1}s_{j+1} + \binom{k+1}{2}s_{j+2} + \cdots + \binom{k+n-j-1}{n-j}s_n \, .$$

One should think of $M^{(k)}$ as the space of k-jets of n-dimensional ordinary integral manifolds of $\mathcal{I}$ in the sense that two ordinary integral manifolds $f : N \to M$ and $g : N \to M$ represent the same k-jet of an integral manifold at $x \in N$ if and only if $f^{(k)}(x) = (g \circ h)^{(k)}(x)$ for some diffeomorphism $h : N \to N$ such that $h(x) = x$.

Note that

$$\dim M^{(k)} = n + \binom{k}{0}s_0 + \binom{k+1}{1}s_1 + \binom{k+2}{2}s_2 + \cdots + \binom{k+n}{n}s_n \, ,$$

which is what one would expect for a 'solution space' that depends on s_0 constants, s_1 functions of 1 variable, s_2 functions of 2 variables, ..., and s_n functions of n variables. See [12] for a further discussion of this point.

4.4. **Prolonging Cartan structure equations.** This idea can also be applied to understanding the differential invariants of the solutions to a system of Cartan structure equations such as (3.6). Starting with these equations, one can augment them with a system for the b^ρ, namely

$$(4.1) \qquad \mathrm{d}b^\rho = \left(G_i^\rho(a,b) + H_{i\tau}^\rho(a,b)c^\tau \right) \omega^i$$

where the functions $H_{i\tau}^\rho$ for $1 \leq \tau \leq \dim A(a,b)^{(1)}$ are a basis for the first prolongation space of the tableau $A(a,b)$, i.e., they give a basis for the solutions of the homogeneous equations

$$\frac{\partial F_i^\alpha}{\partial b^\rho}(a,b)h_j^\rho - \frac{\partial F_j^\alpha}{\partial b^\rho}(a,b)h_i^\rho = 0.$$

Using Cartan's ideas, it is not difficult to show that, if the system (3.6) satisfies the hypotheses of Theorem 3, then the prolonged system of structure equations consisting of (3.6) and (4.1) will also satisfy the hypotheses of Theorem 3 and that the Cartan characters of the tableau of the prolonged system will be

$$s_i^{(1)} = s_i + s_{i+1} + \cdots + s_n \,.$$

In particular, in this case, for any given (a_0, b_0, c_0) there will exist an augmented coframing (a, b, c, ω) satisfying the prolonged structure equations for which (a, b, c) assumes the value (a_0, b_0, c_0).

This leads naturally to the notion of 'differential invariants' for distinguishing augmented coframings (a, b, ω) satisfying (3.6) up to diffeomorphism. Recall that two such coframings (a, b, ω) on M^n and $(\bar{a}, \bar{b}, \bar{\omega})$ on $\bar{M}^n$ are equivalent up to diffeomorphism if there exists a diffeomorphism $h : \bar{M} \to M$ satisfying $(\bar{a}, \bar{b}, \bar{\omega}) = h^*(a, b, \omega)$. Obviously, this will imply that, if $\mathrm{d}b^\rho = b_i^\rho \omega^i$ and $\mathrm{d}\bar{b}^\rho = \bar{b}_i^\rho \bar{\omega}^i$, then $\bar{b}_i^\rho = h^*(b_i^\rho)$ and similarly for all of the derivatives of the b_j^ρ expanded in terms of the ω^i.

Following Cartan's terminology, one often speaks of the a^α as the *primary* (or *fundamental*) invariants of the augmented coframing and the b^ρ and b_i^ρ, etc. as *derived* invariants. (Here 'invariant' means 'invariant under diffeomorphism equivalence'.)

Thus, the import of Theorem 3 is that one sees that, in addition to being able to freely specify the values of the s primary invariants (i.e., the a^α) of an augmented coframing (a, b, ω) satisfying (3.6) at a point, one can also freely specify their first derived invariants (i.e., the b^ρ), which are $r = s_1 + s_2 + \cdots + s_n$ in number, at the point, and freely specify a certain number of second derived invariants (i.e., the c^τ) which are $r^{(1)} = s_1 + 2s_2 + \cdots + ns_n$ in number, at the point, and so on.

Applying prolongations successively, one sees that the number of freely specifiable differential invariants of augmented coframings satisfying (3.6) of derived order less than or equal to k is equal to

$$s + \binom{k}{1}s_1 + \binom{k+1}{2}s_2 + \cdots + \binom{k+n-1}{n}s_n \,.$$

In a sense that can be made precise, this is the dimension of the space of k-jets of diffeomorphism equivalence classes of augmented coframings satisfying (3.6).

It is in this sense that one can assert that, up to diffeomorphism, the 'general' augmented coframing satisfying a given involutive system of Cartan structure

equations depends on s_1 functions of 1 variable, s_2 functions of 2 variables, and so on.

Similar remarks apply to the structure equations of Theorem 4. In fact, the first prolongation of these structure equations yield structure equations to which Theorem 3 applies, so that one could have simply quoted Theorem 3 to prove Theorem 4. This may make the reader wonder why this latter theorem is useful. The reason is this: It is often simpler to check the hypotheses of Theorem 4 for a given set of structure equations than it is to check the hypotheses of Theorem 3 for the prolonged set of structure equations (as the reader will see in the examples).

4.5. **Non-ordinary prolongation and the Cartan-Kuranishi Theorem.** In most cases, $\mathcal{V}_n(\mathcal{I})$ does not consist entirely of ordinary integral elements, and even when the open subset $\mathcal{V}_n^o(\mathcal{I}) \subset \mathcal{V}_n(\mathcal{I})$ is not empty, one is often interested in at least some components of the complement and would like to know when there exist integral manifolds tangent to these non-ordinary integral elements.

Cartan's prescription for treating this situation was to prolong the non-ordinary integral elements as well: Let $M^{(1)} \subset \mathcal{V}_n(\mathcal{I})$ be any submanifold of $\mathcal{V}_n(\mathcal{I})$ (in most applications, it will be a component of a smooth stratum of $\mathcal{V}_n(\mathcal{I})$ that does not lie in $\mathcal{V}_n^o(\mathcal{I})$). Then, again, one can construct the ideal $\mathcal{I}^{(1)}$ generated by the sections of the contact subbundle $C \subset T^*M^{(1)}$ and one can consider $\mathcal{V}_n\big(\mathcal{I}^{(1)}\big)$, looking for ordinary integral elements of this ideal whose projections to M are injective. If one finds them, then one has existence for integral manifolds tangent to these non-regular integral elements. If one does not find them, one can continue the prolongation process as long as it results in ideals that have integral elements.

Cartan believed that iterating this process would always ultimately result in either an ideal with no integral elements of dimension n or else one that had ordinary integral elements. He was never actually able to prove this result, though.

Eventually, a version of this 'prolongation theorem' (in the real analytic category, of course) was proved by Kuranishi [11]. The hypotheses of the Cartan-Kuranishi Prolongation Theorem are somewhat technical, so I refer you to Kuranishi's original paper [11] for those.

In practice, one uses the Prolongation Theorem as a justification for computing successively higher prolongations until one reaches either incompatibility (i.e., the non-existence of integral elements) or involutivity (i.e., the existence of ordinary integral elements), which, nearly always, is what one must do anyway in order to prove existence of solutions via Cartan-Kähler.

5. Some applications

There are many applications of these structure theorems in differential geometry. Here is a sample of such applications meant to give the reader a sense of how they are used in practice. For further applications to differential geometry, the reader can hardly do better than to consult Cartan's own beautiful collection of instructive examples [9].

5.1. Surface metrics with $|\nabla K|^2 = 1$. Consider the metrics whose Gauss curvature satisfies $|\nabla K|^2 = 1$. The structure equations are

$$\begin{aligned}
d\omega_1 &= -\omega_{12} \wedge \omega_2 \\
d\omega_2 &= \omega_{12} \wedge \omega_1 \qquad\qquad \omega_1 \wedge \omega_2 \wedge \omega_{12} \neq 0, \\
d\omega_{12} &= K\,\omega_1 \wedge \omega_2
\end{aligned}$$

where

$$dK = \cos b\,\omega_1 + \sin b\,\omega_2\,.$$

for some function b. (Here, b is the 'free derivative'.)

Now $d^2 = 0$ is an identity for the forms in the coframing $\omega = (\omega_1, \omega_2, \omega_{12})$, while

$$0 = d(dK) = (db - \omega_{12}) \wedge (-\sin b\,\omega_1 + \cos b\,\omega_2).$$

It follows that the hypotheses of Theorem 3 are satisfied, with the characters of the tableau of free derivatives being $s_1 = 1$, $s_2 = s_3 = 0$. Thus, the general (local) solution depends on one function of one variable.

The prolonged system will have

$$db = \omega_{12} + c\,(-\sin b\,\omega_1 + \cos b\,\omega_2)$$

where c is now the new 'free derivative', etc.

(Of course, it is not difficult to integrate the structure equations in this simple case and find an explicit normal form involving one arbitrary function of one variable, but I will leave this to the reader.)

5.2. Surface metrics of Hessian type. Now, an application of Cartan's original theorem. The goal is to study those Riemannian surfaces (M^2, g) whose Gauss curvature K satisfies the second order system

$$\mathrm{Hess}_g(K) = a(K)g + b(K)dK^2$$

for some functions a and b of one variable.

Writing $g = \omega_1{}^2 + \omega_2{}^2$ on the orthonormal frame bundle F^3 of M, the structure equations become

$$\begin{aligned}
d\omega_1 &= -\omega_{12} \wedge \omega_2 & d\omega_{12} &= K\,\omega_1 \wedge \omega_2 \\
d\omega_2 &= \omega_{12} \wedge \omega_1 & dK &= K_1\,\omega_1 + K_2\,\omega_2
\end{aligned}$$

and the condition to be studied is encoded as

$$\begin{pmatrix} dK_1 \\ dK_2 \end{pmatrix} = \begin{pmatrix} -K_2 \\ K_1 \end{pmatrix} \omega_{12} + \begin{pmatrix} a(K) + b(K)\,K_1{}^2 & b(K)\,K_1 K_2 \\ b(K)\,K_1 K_2 & a(K) + b(K)\,K_2{}^2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Applying $d^2 = 0$ to these two equations yields

$$\big(a'(K) - a(K)b(K) + K\big)\,K_i = 0 \qquad \text{for } i = 1, 2.$$

Thus, unless $a'(K) = a(K)b(K) - K$, such metrics have K constant.

Conversely, suppose that $a'(K) = a(K)b(K) - K$. The question becomes 'Does there exist a 'solution' (F^3, ω) to the following system?'

$$\begin{aligned}
d\omega_1 &= -\omega_{12} \wedge \omega_2 \\
d\omega_2 &= \omega_{12} \wedge \omega_1 \qquad\qquad \omega_1 \wedge \omega_2 \wedge \omega_{12} \neq 0, \\
d\omega_{12} &= K\,\omega_1 \wedge \omega_2
\end{aligned}$$

where

$$\begin{pmatrix} \mathrm{d}K \\ \mathrm{d}K_1 \\ \mathrm{d}K_2 \end{pmatrix} = \begin{pmatrix} K_1 & K_2 & 0 \\ a(K) + b(K)\,K_1{}^2 & b(K)\,K_1 K_2 & -K_2 \\ b(K)\,K_1 K_2 & a(K) + b(K)\,K_2{}^2 & K_1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_{12} \end{pmatrix}.$$

Since $\mathrm{d}^2 = 0$ is formally satisfied for these structure equations, Theorem 2 applies and guarantees that, for any constants (k, k_1, k_2), there is a local solution with the invariants (K, K_1, K_2) taking the value (k, k_1, k_2).

In fact, the above equations show that, on a solution, the $\mathbb{R}^3$-valued function (K, K_1, K_2) either has rank 0 (if $K_1 = K_2 = a(K) = 0$) or rank 2. Moreover, one sees that

$$-\big(a(K) + b(K)(K_1{}^2 + K_2{}^2)\big)\,\mathrm{d}K + K_1\,\mathrm{d}K_1 + K_2\,\mathrm{d}K_2 = 0,$$

so that the image of a connected solution lies in an integral leaf of this 1-form, which only vanishes when $K_1 = K_2 = a(K) = 0$. Setting $L = K_1{}^2 + K_2{}^2$, this expression becomes

$$-2\big(a(K) + b(K)L\big)\,\mathrm{d}K + \mathrm{d}L = 0,$$

which has an integrating factor: If $\lambda(K)$ is a nonzero solution to $\lambda'(K) = -b(K)\lambda(K)$, then

$$-2\lambda(K)^2 a(K)\,\mathrm{d}K + \mathrm{d}\big(\lambda(K)^2 L\big) = 0,$$

so that the curvature map has image in a level set of the function $F(K, K_1, K_2) = \lambda(K)^2(K_1{}^2 + K_2{}^2) - \mu(K)$, where $\mu'(K) = 2\lambda(K)^2 a(K)$. (This function has critical points only where $K_1 = K_2 = a(K) = 0$.)

On any solution (F^3, ω), the vector field Y defined by the equations

$$\omega_1(Y) = \lambda(K)K_2, \quad \omega_2(Y) = -\lambda(K)K_1, \quad \omega_{12}(Y) = \lambda(K)a(K),$$

is a symmetry vector field of the coframing (since the Lie derivative of each of ω_1, ω_2, ω_{12} with respect to Y is zero). It is nonvanishing on a solution of rank 2, and, up to constant multiples, it is the unique symmetry vector field of the coframing on any connected solution.

For simplicity, I will only consider the case $b(K) \equiv 0$ in the remainder of this discussion. In this case, $a'(K) = -K$, so $a(K) = \frac{1}{2}(C - K^2)$ for some constant C and $\lambda'(K) = 0$, so one can take $\lambda(K) \equiv 1$.

The most interesting case is when $C > 0$, and, by scaling the metric g by a constant, one can reduce to the case $C = 1$. Thus, the equations simplify to

$$\begin{pmatrix} \mathrm{d}K \\ \mathrm{d}K_1 \\ \mathrm{d}K_2 \end{pmatrix} = \begin{pmatrix} K_1 & K_2 & 0 \\ \frac{1}{2}(1 - K^2) & 0 & -K_2 \\ 0 & \frac{1}{2}(1 - K^2) & K_1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_{12} \end{pmatrix}.$$

and these functions satisfy

$$F(K, K_1, K_2) = K_1{}^2 + K_2{}^2 + \tfrac{1}{3}K^3 - K = C$$

where C is a constant (different from the previous C, which is now normalized to 1).

There are two critical points of F, namely $(K, K_1, K_2) = (\pm 1, 0, 0)$, and these correspond to the surfaces whose Gauss curvature is identically $+1$ or identically -1. These clearly exist globally so it remains to consider the other level sets.

The level sets with $C < -\frac{2}{3}$ are connected and contractible, in fact, they can be written as graphs of K as a function of $K_1{}^2 + K_2{}^2$. $C = -\frac{2}{3}$ contains the critical point $(K, K_1, K_2) = (1, 0, 0)$, but away from this point, it is also a smooth graph. When $-\frac{2}{3} < C < \frac{2}{3}$, the level set has two smooth components, a compact 2-sphere

that encloses the critical point $(1, 0, 0)$ and a graph of K as a smooth function of $K_1{}^2 + K_2{}^2$. The level set $C = \frac{2}{3}$ is singular at the point $(-1, 0, 0)$, but, minus this point, it has two smooth pieces, one bounded and simply connected, and one unbounded and diffeomorphic to $\mathbb{R} \times S^1$. For $C > \frac{2}{3}$, the level set is connected and contractible.

According to the general theory, for each contractible component L of a (smooth part of a) level set $F = C$, there will exist a simply-connected solution manifold (F^3, ω) whose curvature image is L and whose symmetry vector field Y is complete. Moreover, the time-2π-flow of the vector field X_{12} (i.e., the vector field that satisfies $\omega_1(X_{12}) = \omega_2(X_{12}) = 0$ while $\omega_{12}(X_{12}) = 1$) is a symmetry of the coframing ω and hence is the time-T-flow of Y for some $T > 0$. Dividing F by the $\mathbb{Z}$-action that this generates produces a solution manifold $(\bar{F}, \omega)$ that is no longer simply-connected but on which the flow of X_{12} is 2π-periodic, and this is the necessary and sufficient condition that $\bar{F}$ be the oriented orthonormal frame bundle of a Riemannian surface (M^2, g) satisfying the desired equation.

However, for the components of the level sets that are diffeomorphic to the 2-sphere, this global existence result does not generally hold, i.e., the corresponding solution manifold (F^3, ω) need not be the orthonormal frame bundles of complete Riemannian surfaces (M^2, g). I will explain why for the 2-sphere components of the level sets $F = \epsilon^2 - 2/3$ where $\epsilon > 0$ is small.

Suppose that a connected solution manifold (F^3, ω) whose curvature map has, as image, such a 2-sphere component is found and that the symmetry vector field Y as defined above is complete on it. Then the metric $h = \omega_1{}^2 + \omega_2{}^2 + \omega_{12}{}^2$ must be complete on F. Now, for small positive ϵ, one has that K is close to 1 while K_1 and K_2 are close to zero, so it follows from a computation that the sectional curvatures of h are all positive. In particular, the completeness of the metric on F^3 implies, by Bonnet-Meyers, that it is compact, with finite fundamental group.

By passing to a finite cover, one can assume that F is simply connected. I claim that the symmetry vector field Y has closed orbits and that its flow generates an S^1-action on F. To see this, note that the map $(K, K_1, K_2) : F \to \mathbb{R}^3$ submerses onto the 2-sphere leaf. Hence the fibers over the two points where $K_1 = K_2 = 0$ must be a finite collection of circles that are necessarily integral curves of the vector field Y, which has no singular points. In particular, the flow of Y on one of these circles must be periodic, but, because the flow of Y preserves the coframing ω, if some time $T > 0$ flow of Y has a fixed point, then the time T flow of Y must be the identity. Thus, the flow of Y is periodic with some minimal positive period $T > 0$, so it generates a free S^1-action on F. The quotient by this free S^1-action is a connected quotient surface that is a covering of the 2-sphere. Since this covering must be trivial, the orbits of Y are the fibers of the map (K, K_1, K_2) to the 2-sphere. In particular, F, being connected and simply-connected, must be diffeomorphic to the 3-sphere.

Now, consider the vector field X_{12} on F as defined above. This vector field is (K, K_1, K_2)-related to the vector field

$$-K_2 \frac{\partial}{\partial K_1} + K_1 \frac{\partial}{\partial K_2}$$

on $\mathbb{R}^3$ whose flow is rotation about the K-axis with period 2π.

It also follows that the flow of X_{12} preserves the two circles that are defined by $K_1 = K_2 = 0$. If (F, ω) is to be a covering of the orthonormal frame bundle of

a Riemannian surface (M^2, g), then X_{12} must be periodic of period $2k\pi$ for some integer $k > 0$. As already remarked, by the structure equations, the 2π-flow of X_{12}, say Ψ, is a symmetry of the coframing and hence must be the time $R > 0$ flow of Y for some unique $R \in (0, T]$.

Now, along each of the two circles in F defined by $K_1 = K_2 = 0$, one has $Y = a(K)X_{12} \neq 0$. The two points where $K_1 = K_2 = 0$ satisfy $K = K_\pm(\epsilon)$ where $K_-(\epsilon) < 1 < K_+(\epsilon)$ and $\frac{1}{3}K_\pm(\epsilon)^3 - K_\pm(\epsilon) = \epsilon^2 - \frac{2}{3}$. In fact, one finds expansions

$$K_\pm(\epsilon) = 1 \pm \epsilon - \tfrac{1}{6}\epsilon^2 \pm \tfrac{5}{72}\epsilon^3 - \cdots$$

and this implies that

$$a\big(K_\pm(\epsilon)\big) = \tfrac{1}{2}\big(1 - K_\pm(\epsilon)^2\big) = \mp\epsilon - \tfrac{1}{3}\epsilon^2 + \cdots.$$

Thus the ratios of X_{12} to Y on these two circles are not equal or opposite, and hence Y cannot have the same period on these two circles, which is impossible. Thus, there cannot be a global solution surface for such a leaf.

5.3. **Prescribed curvature equations for Finsler surfaces.** For an oriented Finsler surface (M^2, F), Cartan showed that the 'tangent indicatrix' (i.e., the analog of the unit sphere bundle) $\Sigma \subset TM$ carries a canonical coframing $(\omega_1, \omega_2, \omega_3)$ generalizing the case of the unit sphere bundle of a Riemannian metric. It satisfies structure equations

$$(5.1) \qquad \begin{aligned} d\omega_1 &= -\omega_2 \wedge \omega_3 \\ d\omega_2 &= -\omega_3 \wedge \omega_1 - I\,\omega_2 \wedge \omega_3 \qquad\qquad \omega_1 \wedge \omega_2 \wedge \omega_3 \neq 0, \\ d\omega_3 &= -K\,\omega_1 \wedge \omega_2 - J\,\omega_2 \wedge \omega_3 \end{aligned}$$

where I have written ω_3 for what would be $-\omega_{12}$ in the Riemannian case. The functions I, J, and K are the *Finsler structure functions*.

One can check that Theorem 4 applies directly to these equations, with V of dimension 3 and $A \subset V \otimes \Lambda^2(V^*)$ an affine subspace of dimension 3 (and on which I, J, and K are coordinates). The Cartan characters are $s_1 = 0$, $s_2 = 2$, and $s_3 = 1$. Thus, the general Finsler surface depends on one function of 3 variables, which is to be expected, since a Finsler structure on M is locally determined by choosing a hypersurface in TM (satisfying certain local convexity conditions) to be the tangent indicatrix $\Sigma \subset TM$. In fact, if $\omega = (\omega_i)$ is any coframing on a 3-manifold Σ^3 that satisfies (5.1) such that the space M of leaves of the system $\omega_1 = \omega_2 = 0$ can be given the structure of a smooth surface for which the natural projection $\pi : \Sigma \to M$ is a submersion, then Σ has a natural immersion $\iota : \Sigma \to TM$ defined by letting $\iota(u) = \pi'\big(X_1(u)\big)$ for $u \in \Sigma$, where X_1 is the vector field on Σ dual to ω_1, and, locally, this defines a Finsler structure on M.

Taking the exterior derivatives of (5.1), one finds that they satisfy identities (the 'Bianchi identities' of Finsler geometry)

$$(5.2) \qquad \begin{aligned} dI &= & J\,\omega_1 &+ I_2\,\omega_2 &+ I_3\,\omega_3\,, \\ dJ &= -(K_3 + KI)\,\omega_1 &+ J_2\,\omega_2 &+ J_3\,\omega_3\,, \\ dK &= & K_1\,\omega_1 &+ K_2\,\omega_2 &+ K_3\,\omega_3\,. \end{aligned}$$

for seven new functions I_2, I_3, ..., K_3. These are the free derivatives of the structure theory. As expected from the general theory, the tableau of free derivatives of the prolonged system, i.e., (5.1) together with (5.2), is involutive with characters $s_1 = s_2 = 3$ and $s_3 = 1$.

Now, by the structure equations (5.2), if $I = 0$, then $J = 0$ and $K_3 = 0$, so that the Bianchi identities reduce to

$$\mathrm{d}K = K_1\,\omega_1 + K_2\,\omega_2\,,$$

which is simply the Riemannian case. Note that in this case, the tableau of free derivatives has $s_1 = s_2 = 1$ while $s_3 = 0$, corresponding to the fact that Riemannian surfaces depend locally on one function of 2 variables (up to diffeomorphism).

One can, of course, study other curvature conditions. For example, the *Landsberg* surfaces are those for which $J = 0$. They satisfy structure equations

$$
\begin{aligned}
\mathrm{d}I &= 0\,\omega_1 + I_2\,\omega_2 + I_3\,\omega_3\,, \\
\mathrm{d}K &= K_1\,\omega_1 + K_2\,\omega_2 - KI\,\omega_3\,.
\end{aligned}
$$
(5.3)

The tableau of free derivatives now has $s_1 = s_2 = 2$ and $s_3 = 0$, so that the general Landsberg metric depends on 2 functions of 2 variables. (By the way, this is only a 'microlocal' description of the solutions; constructing global solutions is much more difficult. However, it does suffice to show how 'flexible' the 'microlocal' solutions are.)

Another common curvature condition is the 'K-basic' condition, i.e., when, K, the Finsler-Gauss curvature, is constant on the fibers of the projection $\Sigma \to M$. This is the condition $K_3 = 0$, so that the structure equations become

$$
\begin{aligned}
\mathrm{d}I &= J\,\omega_1 + I_2\,\omega_2 + I_3\,\omega_3\,, \\
\mathrm{d}J &= -KI\,\omega_1 + J_2\,\omega_2 + J_3\,\omega_3\,, \\
\mathrm{d}K &= K_1\,\omega_1 + K_2\,\omega_2 + 0\,\omega_3\,.
\end{aligned}
$$
(5.4)

The tableau of free derivatives now has $s_1 = s_2 = 3$ and $s_3 = 0$, showing that these Finsler structures depend on 3 functions of 2 variables.

Even more restrictive are the Finsler metrics with constant K. These satisfy

$$
\begin{aligned}
\mathrm{d}I &= J\,\omega_1 + I_2\,\omega_2 + I_3\,\omega_3\,, \\
\mathrm{d}J &= -KI\,\omega_1 + J_2\,\omega_2 + J_3\,\omega_3\,, \\
\mathrm{d}K &= 0\,\omega_1 + 0\,\omega_2 + 0\,\omega_3\,.
\end{aligned}
$$
(5.5)

The tableau of free derivatives now has $s_1 = s_2 = 2$ and $s_3 = 0$, showing that these Finsler structures depend on 2 functions of 2 variables. (For those who know about characteristics, note that, in this case, a covector $\xi = \xi_1\,\omega_1 + \xi_2\,\omega_2 + \xi_3\,\omega_3$ is characteristic for this tableau if and only if $\xi_1 = 0$. Thus, the 'arbitrary functions' are actually functions on the leaf space of the geodesic flow $\omega_2 = \omega_3 = 0$. This suggests (and, of course, it turns out to be true) that these structures are actually geometric structures on the space of geodesics in disguise.)

5.4. Ricci-gradient metrics in dimension 3. Here are some sample problems from Riemannian geometry. In the following, for simplicity of notation, I will consider only the 3-dimensional case, but the higher dimensional cases are not much different.

Consider the problem of studying those Riemannian manifolds (M, g) for which there exists a function f such that $\mathrm{Ric}(g) = (\mathrm{d}f)^2 + H(f)\,g$, where H is a specified function of one variable. Most metrics g will not have such a 'Ricci potential', and it is not immediately apparent how many such metrics there are.

The problem can be set up in structure equations as follows: On the orthonormal frame bundle $F^6 \to M^3$ of g, one has the usual first structure equations

$$\text{(5.6)} \qquad \mathrm{d}\omega_i = -\omega_{ij} \wedge \omega_j$$

and the second structure equations (in dimension 3) can be written in the form

$$\text{(5.7)} \qquad \begin{pmatrix} \mathrm{d}\omega_{23} \\ \mathrm{d}\omega_{31} \\ \mathrm{d}\omega_{12} \end{pmatrix} = - \begin{pmatrix} \omega_{12} \wedge \omega_{31} \\ \omega_{23} \wedge \omega_{12} \\ \omega_{31} \wedge \omega_{23} \end{pmatrix} - \left(R - \tfrac{1}{2}\mathrm{tr}(R)\, I_3 \right) \begin{pmatrix} \omega_2 \wedge \omega_3 \\ \omega_3 \wedge \omega_1 \\ \omega_1 \wedge \omega_2 \end{pmatrix}$$

where $R = (R_{ij})$ is the symmetric matrix of the Ricci tensor. By hypothesis, there exists a function f such that

$$R_{ij} = f_i f_j + H(f)\delta_{ij}$$

where

$$\text{(5.8)} \qquad \mathrm{d}f = f_1\, \omega_1 + f_2\, \omega_2 + f_3\, \omega_3\,.$$

The four functions (f, f_1, f_2, f_3) will play the role of the a^α in the structure equations. Since $\mathrm{d}(\mathrm{d}f) = 0$, there exist functions $f_{ij} = f_{ji}$ such that

$$\text{(5.9)} \qquad \mathrm{d}f_i = -\omega_{ij}f_j + f_{ij}\,\omega_j\,.$$

The symmetry of R implies that the equations $\mathrm{d}(\mathrm{d}\omega_i) = 0$ are identities, but, when one computes $\mathrm{d}(\mathrm{d}\omega_{ij}) = 0$, one finds that these relations can be written as

$$\left(2(f_{11} + f_{22} + f_{33}) - H'(f) \right)\mathrm{d}f = 0.$$

Thus, either $\mathrm{d}f = 0$, in which case f is constant (so that the metric is Einstein), or else the relation

$$f_{11} + f_{22} + f_{33} - \tfrac{1}{2}H'(f) = 0$$

must hold. So impose this condition, and rewrite the above equation in the form

$$\text{(5.10)} \qquad \mathrm{d}f_i = -\omega_{ij}f_j + \left(b_{ij} + \tfrac{1}{6}H'(f)\delta_{ij} \right)\omega_j\,.$$

where the (new) $b_{ij} = b_{ji}$ are subject to the trace condition $b_{11} + b_{22} + b_{33} = 0$. These b_{ij} will play the role of the b^ρ in the structure equations.

Thus, the problem can be thought of as seeking coframings $\omega = (\omega_i, \omega_{ij})$ and functions (f, f_i) on a 6-manifold F^6 that satisfy the equations (5.6), (5.7), (5.8), and (5.10), where the $b_{ij} = b_{ji}$ are subject to $b_{11} + b_{22} + b_{33} = 0$.

The tableau of the free derivatives is involutive, with characters $s_1 = 3$, $s_2 = 2$, and $s_k = 0$ for $3 \le k \le 6$. Moreover, the equations $\mathrm{d}(\mathrm{d}\omega_i) = \mathrm{d}(\mathrm{d}\omega_{ij}) = \mathrm{d}(\mathrm{d}f) = 0$ are identities while the equations $\mathrm{d}(\mathrm{d}f_i) = 0$ are satisfiable in the form

$$\mathrm{d}b_{ij} = -b_{ik}\omega_{kj} - b_{kj}\omega_{ki} + F\left(3f_i\omega_j + 3f_j\omega_i - 2\delta_{ij}f_k\omega_k \right) + b_{ijk}\omega_k$$

where $F = \tfrac{1}{10}\left(f_1{}^2 + f_2{}^2 + f_3{}^2 + H(f) + \tfrac{1}{3}H''(f) \right)$ and where $b_{ijk} = b_{jik} = b_{ikj}$ and $b_{iik} = 0$. Hence, there are $7 = s_1 + 2\,s_2 + \cdots + 6\,s_6$ independent free derivatives of the b_{ij}, the maximum allowed by the characters of their tableau.

Thus, the hypotheses of Theorem 3 are satisfied. Consequently, when H is an analytic function, the pairs (g, f) that satisfy $\mathrm{Ric}(g) = (\mathrm{d}f)^2 + H(f)\, g$ depend on 2 functions of 2 variables (up to diffeomorphism).

(For those who know about the characteristic variety: A nonzero covector $\xi = \xi_i\,\omega_i + \xi_{ij}\,\omega_{ij}$ is characteristic if and only if $\xi_{ij} = 0$ and $\xi_1{}^2 + \xi_2{}^2 + \xi_3{}^2 = 0$. Thus, the real characteristic variety is empty, so the solutions are all real analytic when H is real analytic.)

More generally, one can consider the problem of studying those Riemannian manifolds (M, g) for which there exists a function f such that

$$\text{(5.11)} \qquad \text{Ric}(g) = a(f)\,\text{Hess}_g(f) + b(f)\,(\mathrm{d}f)^2 + c(f)\,g$$

where a, b, and c are specified functions of one variable and $\text{Hess}_g(f) = \nabla\nabla f$ is the Hessian of f with respect to g, i.e., the quadratic form that is the second covariant derivative of f with respect to the Levi-Civita connection of g. For example, when $a(f) = -1$, $b(f) = 0$, and $c(f) = \lambda$ (a constant), (5.11) is the equation for a gradient Ricci soliton. For simplicity, in what follows, I will assume that a, b, and c are real-analytic functions.

If $a(f) \equiv b(f) \equiv 0$, then (5.11) implies that g is an Einstein metric, and so the only solutions (g, f) are ones for which $c(f)$ is a constant. In particular, if $c'(f)$ is not identically vanishing, then the only solutions (g, f) are when g is Einstein and f is a constant.

If $a(f) \equiv 0$ and $b(f) > 0$, one can reduce (5.11) to the case $b(f) \equiv 1$ (which was treated above) by replacing (g, f) by $(g, \phi(f))$, where $\phi'(f)^2 = b(f)$. (Meanwhile, when $b(f) < 0$, one can reduce to $b(f) \equiv -1$ by replacing (g, f) by $(g, \phi(f))$ where $\phi'(f)^2 = -b(f)$. The reader can easily check that the local analysis of this case is essentially the same as the case $a(f) \equiv 0$ and $b(f) \equiv 1$, with a few sign changes.)

In the 'generic' case, in which a is nonvanishing, one can reduce to the case $b(f) \equiv 0$ by replacing (g, f) by $(g, \phi(f))$ where ϕ is a function that satisfies $\phi'(x) > 0$ and $\phi''(x) = (b(x)/a(x))\phi'(x)$. Hence, I will consider only the case $b(f) \equiv 0$ in the remainder of this discussion.

Thus, the equation to be studied is encoded with the same structure equations (5.6), (5.7), and (5.9) but now with the relations

$$R_{ij} = a(f)\,f_{ij} + c(f)\,\delta_{ij}\,,$$

where a is a nonvanishing function. The equations $\mathrm{d}(\mathrm{d}\omega_{ij}) = \mathrm{d}(\mathrm{d}f_i) = 0$ then turn out to imply the relation

$$\mathrm{d}\left(\frac{L(f)}{a(f)}\right) + \left(1 + \frac{a'(f)}{a(f)^2}\right)\,\mathrm{d}H(f) + \frac{\big(2a(f)c(f) - c'(f)\big)}{a(f)^2}\,\mathrm{d}f = 0$$

where $L(f) = f_{11} + f_{22} + f_{33}$ and $H(f) = f_1{}^2 + f_2{}^2 + f_3{}^2$. Taking the exterior derivative of this relation yields

$$\left(\frac{a(f)a''(f) - 2a'(f)^2}{a(f)^3}\right)\,\mathrm{d}f \wedge \mathrm{d}H(f) = 0.$$

At this point, the study of these equations divides into cases, depending on whether $aa'' - 2(a')^2$ vanishes identically or not.

If the function $aa'' - 2(a')^2$ does not vanish identically, then any pair (g, f) that satisfies the original equation must also satisfy equations of the form

$$f_1{}^2 + f_2{}^2 + f_3{}^2 = h(f)$$

and

$$f_{11} + f_{22} + f_{33} = a(f)l(f)$$

for functions l and h of a single variable that satisfy

$$l'(x) + \left(1 + \frac{a'(x)}{a(x)^2}\right)h'(x) + \frac{\big(2a(x)c(x) - c'(x)\big)}{a(x)^2} = 0.$$

Upon differentiation, the first of these equations implies

$$2 f_{ij} f_j = h'(f) f_i$$

which, as long as $h(f) > 0$, gives three equations on the free derivatives $f_{ij} = f_{ji}$. Moreover, the equation $f_{11} + f_{22} + f_{33} = a(f) l(f)$ is independent from these three. This means that there is only a 2-parameter family of possible variation in the f_{ij}. In fact, the tableau of free derivatives in this case is involutive with $s_1 = 2$ and all $s_i = 0$ for $i > 1$, so that solutions of this system depend on at most[14] two functions of one variable (three if you count the function h). Thus, the pairs (g, f) that satisfy the above equation are rather rigid.

On the other hand, if $aa'' - 2(a')^2$ vanishes identically, then $a(f) = 1/(c_0 + c_1 f)$ for some constants c_0 and c_1, not both zero.

If $c_1 = 0$, then, by scaling f, one can reduce to the case $a(f) = 1$ and the original equation becomes

$$R_{ij} = f_{ij} + c(f) \, \delta_{ij} \,,$$

while the relation above becomes

$$f_{11} + f_{22} + f_{33} + f_1{}^2 + f_2{}^2 + f_3{}^2 - c(f) + 2C(f) = \lambda,$$

where $C'(f) = c(f)$, and where λ is a constant. Adding this relation on the 'free derivatives' f_{ij} yields a tableau of free derivatives that has $s_1 = 3$, $s_2 = 2$ and $s_j = 0$ for $j > 2$. Moreover, a short calculation reveals that this relation satisfies the conditions of Theorem 3, so, up to diffeomorphism, the local general pairs (g, f) that satisfy a relation of the form $\mathrm{Ric}(g) = \mathrm{Hess}_g(f) + c(f)g$ (for a fixed real-analytic function $c(f)$) depend on two functions of two variables.

Meanwhile, if $c_1 \neq 0$, then by translating and scaling f, one can reduce to the case $a(f) = 1/f$, and one gets a similar result, that, up to diffeomorphism, the local general pairs (g, f) (with, say $f > 0$) that satisfy a relation of the form $\mathrm{Ric}(g) = (\mathrm{Hess}_g(f))/f + c(f)g$ (for a fixed real-analytic function $c(f)$) also depend on two functions of two variables.

5.5. Riemannian 3-manifolds with constant Ricci eigenvalues.
In dimension 3, a different way of writing the structure equations on the orthonormal frame bundle F^6 of (M^3, g) is to write them in 'vector' form as

$$\mathrm{d}\eta = -\theta \wedge \eta \tag{5.12}$$

and

$$\mathrm{d}\theta = -\theta \wedge \theta + \left(R - \tfrac{1}{4} \mathrm{tr}(R) I_3 \right) \eta \wedge {}^t\eta + \eta \wedge {}^t\eta \left(R - \tfrac{1}{4} \mathrm{tr}(R) I_3 \right) \tag{5.13}$$

where $\eta = (\eta_i)$ takes values in $\mathbb{R}^3$ (thought of as columns of real numbers of height 3) and $\pi^* g = {}^t\eta \circ \eta$, while $\theta = -{}^t\theta = (\theta_{ij})$ takes values in $\mathfrak{so}(3)$, the space of skewsymmetric 3-by-3 matrices, and $R = {}^tR$ is the 3-by-3 symmetric matrix that represents the Ricci curvature, i.e., $R = (R_{ij})$ and $\pi^* \big(\mathrm{Ric}(g) \big) = R_{ij} \, \eta_i \circ \eta_j$.

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[14]The reason for the 'at most' is that I have not verified that the torsion is absorbable, so I cannot claim that this prolonged system is involutive.

5.5.1. *The general metric.* Setting $V = \mathbb{R}^3 \oplus \mathfrak{so}(3)$ (so that, again, $n = 6$), then $\omega = (\eta, \theta)$ is a V-valued coframing, and the above structure equations take the form $\mathrm{d}\omega = -\frac{1}{2}C(\omega \wedge \omega)$, where C takes values in a 6-dimensional affine subspace $A \subset V \otimes \Lambda^2(V^*)$.

The exterior derivatives of these structure equations then give the compatibility conditions: One has $\mathrm{d}(\mathrm{d}\eta) = 0$, and, setting $\rho = \mathrm{d}R + \theta R - R\theta$, one finds

$$(5.14) \qquad \mathrm{d}(\mathrm{d}\theta) = \left(\rho - \tfrac{1}{4}\operatorname{tr}(\rho)I_3\right) \wedge \eta \wedge {}^t\eta + \eta \wedge {}^t\eta \wedge \left(\rho - \tfrac{1}{4}\operatorname{tr}(\rho)I_3\right),$$

so $A \subset V \otimes \Lambda^2(V^*)$, an affine subspace, is a Jacobi manifold. Inspection shows that its tableau has characters $s_0 = s_1 = 0$, $s_2 = s_3 = 3$, and $s_k = 0$ for $k = 4, 5, 6$. Now, the three 3-forms $\mathrm{d}(\mathrm{d}\theta)$ place 21 restrictions on the 36 coefficients of $S \in \operatorname{Hom}(V, \mathbb{R}^6)$ in order that the equation $\rho - S(\eta, \theta) = 0$ should define an integral element. Since $21 = c_0 + c_1 + c_2 + c_3 + c_4 + c_5$, it follows that the tableau is involutive, so that A is an involutive Jacobi manifold.

Thus, Theorem 4 yields the expected result that the general metric in dimension 3 modulo diffeomorphism depends on 3 functions of 2 variables and 3 functions of 3 variables. Applying prolongation to the structure equations would yield that the number of differential invariants of the coframing of order at most $k + 1$ is

$$\sum_{j=0}^{6} \binom{k+j-1}{j} s_j = \frac{k(k+1)(k+5)}{2},$$

which is the classically known number of independent derivatives of the curvature functions R_{ij} of order at most $k-1$ (as expected, since the R_{ij} themselves are the first derivatives of the coframing ω).

5.5.2. *Constant Ricci eigenvalues.* More interesting are the proper submanifolds of A that are involutive Jacobi manifolds. For example, suppose that one wanted to determine the generality (modulo diffeomorphisms) of the space of metrics whose Ricci tensor has constant eigenvalues. Thus, one takes the above structure equations and imposes that

$$(5.15) \qquad\qquad\qquad R = {}^tPCP = P^{-1}CP,$$

where C is a constant diagonal matrix with diagonal entries $c = (c_1, c_2, c_3)$ where $c_1 \geq c_2 \geq c_3$ and P lies in $\mathrm{SO}(3)$. Restricting R to take this form in the structure equations defines a (non-affine) submanifold $B_c \subset A \subset V \otimes \Lambda^2(V^*)$ that has dimension 3 (and is diffeomorphic to the quotient of $\mathrm{SO}(3)$ by its diagonal subgroup) when the c_i are distinct, dimension 2 (and is diffeomorphic to $\mathbb{RP}^2$) when two of the c_i are equal, and has dimension 0 (and is a single point) when the c_i are all equal.

One can write the structure equations in a relatively uniform way by setting $\bar{\eta} = P\eta$ and $\pi = \mathrm{d}PP^{-1} - P\theta P^{-1} = -{}^t\pi$, for then the above equations can be written

$$0 = P\mathrm{d}(\mathrm{d}\theta)P^{-1} = (C\pi - \pi C) \wedge \bar{\eta} \wedge {}^t\bar{\eta} + \bar{\eta} \wedge {}^t\bar{\eta} \wedge (C\pi - \pi C)$$

and the three 3-forms in the skew-symmetric matrix on the righthand side of this equation are seen to be

$$\Upsilon_1 = \left((c_3-c_1)\pi_2 \wedge \bar{\eta}_2 - (c_1-c_2)\pi_3 \wedge \bar{\eta}_3\right) \wedge \bar{\eta}_1$$
$$\Upsilon_2 = \left((c_1-c_2)\pi_3 \wedge \bar{\eta}_3 - (c_2-c_3)\pi_1 \wedge \bar{\eta}_1\right) \wedge \bar{\eta}_2$$
$$\Upsilon_3 = \left((c_2-c_3)\pi_1 \wedge \bar{\eta}_1 - (c_3-c_1)\pi_2 \wedge \bar{\eta}_2\right) \wedge \bar{\eta}_3$$

where $\pi = (\pi_{ij}) = (\epsilon_{ijk}\pi_k)$. Note that the 1-forms π_1, π_2, π_3 complete the components of θ and η to a basis on the frame bundle cross SO(3).

In particular, this formula yields that B_c is a Jacobi manifold for any choice of $c = (c_1, c_2, c_3)$ and that its tableau has rank 3 when the c_i are distinct, rank 2 when exactly two of the c_i are equal, and rank 0 when all of the c_i are equal.

When all of the c_i are equal, the tableau is trivial, and so there is a regular flag (with characters $s_i = 0$) by definition.

5.5.3. *Three distinct, constant eigenvalues.*

When the c_i are distinct, one sees that there is a regular flag for the integral elements described by $\pi_i = 0$ with characters $s_2 = 3$ and $s_i = 0$ otherwise. In fact, a hyperplane in this integral element fails to be the end of a regular flag if and only if it is described by an equation of the form $\xi = \xi_1 \bar{\eta}_1 + \xi_2 \bar{\eta}_2 + \xi_3 \bar{\eta}_3 = 0$ with $\xi_1 \xi_2 \xi_3 = 0$. Consequently, Theorem 4 applies, and it follows that, up to diffeomorphism, Riemannian 3-manifolds with distinct constant eigenvalues of the Ricci tensor depend on 3 arbitrary functions of 2 variables.

5.5.4. *Two distinct, constant eigenvalues.*

However, when exactly two of the c_i are equal, there is no regular flag: One easily checks that the codimensions of the polar spaces of a generic flag for this tableau are $c_0 = c_1 = 0$, while $c_k = 2$ for $k \geq 2$. However, the codimension of the space of integral elements is $9 > c_0 + c_1 + c_2 + c_3 + c_4 + c_5$, as the reader can check. Thus, when two of the c_i are equal, the 2-dimensional Jacobi manifold B_c is not involutive.

This does not mean that there are not Riemannian metrics for which the Ricci tensor has two distinct, constant eigenvalues. To check this, though, one must prolong the structure equations and use Theorem 3 instead of Theorem 4 as follows:

Suppose that $\mathrm{Ric}(g) = {}^t\eta \circ R \circ \eta$ has two distinct constant eigenvalues, say $c_1 \neq c_2$ and c_2 (of multiplicity 2). This means that there is a circle bundle F^4 over M^3 consisting of the g-orthonormal coframes such that $\mathrm{Ric}(g) = c_1\,\eta_1{}^2 + c_2\left(\eta_2{}^2 + \eta_3{}^2\right)$. As the reader can check, this implies that the structure equations on F can be written in the form

$$(5.16) \qquad \begin{aligned} \mathrm{d}\eta_1 &= -2a_1\,\eta_2 \wedge \eta_3 \\ \mathrm{d}\eta_2 &= -\eta_{23} \wedge \eta_3 - \left(a_2\,\eta_2 + (a_1{+}a_3)\,\eta_3\right) \wedge \eta_1 \\ \mathrm{d}\eta_3 &= \eta_{23} \wedge \eta_2 + \left((a_1{-}a_3)\,\eta_2 + a_2\,\eta_3\right) \wedge \eta_1 \\ \mathrm{d}\eta_{23} &= c_2\,\eta_2 \wedge \eta_3 \end{aligned}$$

where a_1, a_2, a_3 are functions satisfying $a_1{}^2 - a_2{}^2 - a_3{}^2 = \tfrac{1}{2}c_1$ and the relations

$$(5.17) \qquad \begin{aligned} \mathrm{d}a_1 &= & & & 2b_3\,\eta_2 &+ & 2b_4\,\eta_3 \\ \mathrm{d}a_2 &= & -\,2a_3\,\eta_{23} &+ & (b_4{+}b_1)\,\eta_2 &+ & (b_3{+}b_2)\,\eta_3 \\ \mathrm{d}a_3 &= & 2a_2\,\eta_{23} &- & (b_3{-}b_2)\,\eta_2 &+ & (b_4{-}b_1)\,\eta_3 \end{aligned}$$

for some functions b_1, b_2, b_3, and b_4.

Conversely, suppose that, on a 4-manifold F, one has an augmented coframing

$$(a, \eta) = (a_1, a_2, a_3, \eta_1, \eta_2, \eta_3, \eta_{32})$$

that satisfies (5.16), (5.17), and $a_1{}^2 - a_2{}^2 - a_3{}^2 = \tfrac{1}{2}c_1$. Then $g = \eta_1{}^2 + \eta_2{}^2 + \eta_3{}^2$ is a metric on the 3-dimensional space M of leaves of $\eta_1 = \eta_2 = \eta_3 = 0$ that satisfies $\mathrm{Ric}(g) = c_1\,\eta_1{}^2 + c_2\left(\eta_2{}^2 + \eta_3{}^2\right)$.

Now, because $\mathrm{d}(a_1{}^2 - a_2{}^2 - a_3{}^2) = \frac{1}{2}\mathrm{d}(c_1) = 0$, the b_i must satisfy the relations

$$a_2\,b_1 \;\; + \;\; a_3\,b_2 \;\; - \;\; (a_3{+}2a_1)\,b_3 \;\; + \;\; a_2\,b_4 = 0,$$
$$-a_3\,b_1 \;\; + \;\; a_2\,b_2 \;\; + \;\; a_2\,b_3 \;\; + \;\; (a_3{-}2a_1)\,b_4 = 0,$$

so that there are really only two 'free derivatives' among the b_i, as these two relations are independent except when $(a_1, a_2, a_3) = (0, 0, 0)$ (and this can only happen if $c_1 = 0$; but when $c_1 = 0$, I will remove the locus where the a_i all vanish from further consideration).

The reader can check that there exist 1-forms $\beta_i \equiv \mathrm{d}b_i \mod \{\eta_1, \eta_2, \eta_3, \eta_{23}\}$ such that

$$a_2\,\beta_1 \;\; + \;\; a_3\,\beta_2 \;\; - \;\; (a_3{+}2a_1)\,\beta_3 \;\; + \;\; a_2\,\beta_4 = 0,$$
$$-a_3\,\beta_1 \;\; + \;\; a_2\,\beta_2 \;\; + \;\; a_2\,\beta_3 \;\; + \;\; (a_3{-}2a_1)\,\beta_4 = 0,$$

and such that the relations

$$\mathrm{d}(\mathrm{d}a_1) \equiv \qquad\qquad 2\beta_3 \wedge \eta_2 \;\; + \;\; 2\beta_4 \wedge \eta_3$$
$$\mathrm{d}(\mathrm{d}a_2) \equiv \;\; + \;\; (\beta_4{+}\beta_1) \wedge \eta_2 \;\; + \;\; (\beta_3{+}\beta_2) \wedge \eta_3$$
$$\mathrm{d}(\mathrm{d}a_3) \equiv \;\; - \;\; (\beta_3{-}\beta_2) \wedge \eta_2 \;\; + \;\; (\beta_4{-}\beta_1) \wedge \eta_3$$

are identities modulo the above structure equations.

Meanwhile, the tableau of free derivatives is involutive, with $s_1 = 2$ and $s_i = 0$ for $i > 2$. Thus, Theorem 3 applies, and one sees that the general such metric depends on 2 functions of 1 variable.

(For those who know about the characteristic variety, one can compute that a covector is characteristic iff it is of the form $\xi = \xi_2\,\eta_2 + \xi_3\,\eta_3$ where (ξ_2, ξ_3) satisfy

$$(a_1{+}a_3)\,\xi_2{}^2 - 2a_2\,\xi_2\xi_3 + (a_1{-}a_3)\,\xi_3{}^2 = 0$$

In particular, the characteristic variety consists of two complex conjugate points when $c_1 > 0$, a double point when $c_1 = 0$, and two real distinct points when $c_1 < 0$. Consequently, the metrics with $c_1 > 0$ will be real-analytic in harmonic coordinates.)

5.6. H-structures with prescribed geometry.
This last set of examples are applications to the geometry of H-structures.

5.6.1. Torsion-free H-structures.
Let $\mathfrak{m}$ be a vector space over $\mathbb{R}$ of dimension m, and let $H \subset \mathrm{GL}(\mathfrak{m})$ be a connected Lie subgroup of dimension r with Lie algebra $\mathfrak{h} \subset \mathfrak{gl}(\mathfrak{m}) = \mathfrak{m} \otimes \mathfrak{m}^*$.

One is interested in determining the generality, modulo diffeomorphism, of the (local) H-structures that are torsion-free, and, more generally, of torsion-free connections on m-manifolds with holonomy contained in (a conjugate of) H.

Remark 14. When the first prolongation space of $\mathfrak{h}$ vanishes, i.e., when

$$\mathfrak{h}^{(1)} = (\mathfrak{h} \otimes \mathfrak{m}^*) \cap (\mathfrak{m} \otimes S^2(\mathfrak{m}^*)) = (0),$$

these two questions are essentially the same, since, in this case, an H-structure that is torsion-free has a unique compatible torsion-free connection, while a torsion-free connection on M whose holonomy is conjugate to a subgroup $K \subset H$ defines an P/N-parameter family of torsion-free H-structures, where $P \subset \mathrm{GL}(\mathfrak{m})$ is the group of elements $p \in \mathrm{GL}(\mathfrak{m})$ such that $p^{-1}Kp \subset H$, while $N \subset H$ is the group of elements such that $p^{-1}Kp = K$.

Now, the geometric objects being studied are the H-structures $\pi : B \to M^m$ endowed with a torsion-free compatible connection. Letting $\eta : TB \to \mathfrak{m}$ be the canonical $\mathfrak{m}$-valued 1-form on B, the torsion-free compatible connection defines an $\mathfrak{h}$-valued 1-form $\theta : TB \to \mathfrak{h}$ satisfying the *first structure equation*

$$\text{(5.18)} \qquad \mathrm{d}\eta = -\theta \wedge \eta$$

and having the equivariance $R_h^*(\theta) = \mathrm{Ad}(h^{-1})\big(\theta\big)$ for all $h \in H$.

One then has the *second structure equation*

$$\text{(5.19)} \qquad \mathrm{d}\theta = -\theta \wedge \theta + \tfrac{1}{2}\,R(\eta \wedge \eta)$$

for a unique *curvature function* $R : B \to \mathfrak{h} \otimes \Lambda^2(\mathfrak{m}^*)$.

Conversely, any manifold B endowed with a coframing

$$\omega = (\eta, \theta) : TB \to \mathfrak{m} \oplus \mathfrak{h} = V$$

satisfying the equations (5.18) and (5.19) for some function $R : B \to \mathfrak{h} \otimes \Lambda^2(\mathfrak{m}^*)$ is locally diffeomorphic to the canonical coframing constructed above from the data of an H-structure on a manifold M endowed with a compatible, torsion-free connection.

Now, because $\mathrm{d}(\mathrm{d}\eta) = 0$, the function R satisfies the *first Bianchi identity*,

$$0 = \mathrm{d}(\mathrm{d}\eta) = -\mathrm{d}\theta \wedge \eta + \theta \wedge \mathrm{d}\eta = -(\mathrm{d}\theta + \theta \wedge \theta) \wedge \eta = -\tfrac{1}{2}\,R(\eta \wedge \eta) \wedge \eta = 0.$$

I.e., R takes values in the kernel $K_0(\mathfrak{h}) \subset \mathfrak{h} \otimes \Lambda^2(\mathfrak{m}^*)$ of the natural map

$$\mathfrak{h} \otimes \Lambda^2(\mathfrak{m}^*) \subset \mathfrak{m} \otimes \mathfrak{m}^* \otimes \Lambda^2(\mathfrak{m}^*) \to \mathfrak{m} \otimes \Lambda^3(\mathfrak{m}^*).$$

(This is the algebraic content of the first Bianchi identity.)

In particular, the combined structure equations (5.18) and (5.19) define a system of equations for the coframing $\omega = (\eta, \theta)$ taking values in $V = \mathfrak{m} \oplus \mathfrak{h}$ for which the structure function is required to take values in an affine space $A_{\mathfrak{h}} \subset V \otimes \Lambda^2(V^*)$ that is modeled on the linear subspace $K_0(\mathfrak{h}) \subset \mathfrak{h} \otimes \Lambda^2(\mathfrak{m}^*) \subset V \otimes \Lambda^2(V^*)$.

Differentiating (5.19) yields, after some algebra, the *second Bianchi identity*

$$0 = \mathrm{d}(\mathrm{d}\theta) = \tfrac{1}{2}\big(\mathrm{d}R + \rho_0'(\theta)R\big)(\eta \wedge \eta),$$

where $\rho_0 : H \to \mathrm{GL}\big(K_0(\mathfrak{h})\big)$ is the induced representation of H on $K_0(\mathfrak{h})$, and $\rho_0' : \mathfrak{h} \to \mathfrak{gl}\big(K_0(\mathfrak{h})\big)$ is the induced map on Lie algebras. This means that

$$\mathrm{d}R = -\rho_0'(\theta)R + R'(\eta),$$

where $R' : B \to K_0(\mathfrak{h}) \otimes \mathfrak{m}^*$ takes values in the kernel $K_1(\mathfrak{h}) \subset K_0(\mathfrak{h}) \otimes \mathfrak{m}^*$ of the natural linear mapping defined by skew-symmetrization

$$K_0(\mathfrak{h}) \otimes \mathfrak{m}^* \subset \mathfrak{h} \otimes \Lambda^2(\mathfrak{m}^*) \otimes \mathfrak{m}^* \to \mathfrak{h} \otimes \Lambda^3(\mathfrak{m}^*).$$

This is the algebraic content of the second Bianchi identity.

In particular, $A_{\mathfrak{h}}$ is a Jacobi manifold, and it is natural to ask when it is involutive, which is a condition on the Lie algebra $\mathfrak{h} \subset \mathfrak{gl}(\mathfrak{m})$. In fact, the test for involutivity is quite simple in this case: One computes the characters s_i of $K_0(\mathfrak{h})$ considered as a tableau in $\mathfrak{h} \otimes \Lambda^2(\mathfrak{m}^*)$. Then Cartan's Bound implies that

$$\dim K_1(\mathfrak{h}) \le s_1 + 2\,s_2 + \cdots + m\,s_m$$

with equality if and only if $K_0(\mathfrak{h})$ and (consequently) $A_{\mathfrak{h}}$ are involutive. Thus, this is a purely algebraic calculation.

Example 4 (Riemannian metrics). In the case that $H = \mathrm{SO}(m)$, the structure equations take the familiar form

$$\mathrm{d}\eta_i = -\theta_{ij} \wedge \eta_j$$

with $\theta_{ij} = -\theta_{ji}$ satisfying

$$\mathrm{d}\theta_{ij} = -\theta_{ik} \wedge \theta_{kj} + \tfrac{1}{2}R_{ijkl}\,\eta_k \wedge \eta_l\,,$$

where the components of the Riemann curvature function R_{ijkl} satisfy the familiar relations $R_{ijkl} = -R_{jikl} = -R_{ijlk}$ and $R_{ijkl} + R_{iklj} + R_{iljk} = 0$. For the tableau $K_0\big(\mathfrak{so}(m)\big)$, the character s_p when $1 \le p \le m$ is the number of independent quantities R_{ijkp} subject to the above relations that have $1 \le k < p$, which one finds to be

$$s_p = \tfrac{1}{2}\,m(p-1)(m-p+1).$$

(Of course, $s_p = 0$ for $m < p < \tfrac{1}{2}m(m{+}1)$.) As expected,

$$s_1 + \cdots + s_m = \tfrac{1}{12}m^2(m^2{-}1) = \dim K_0\big(\mathfrak{so}(m)\big)$$

and one also finds

$$s_1 + 2\,s_2 + \cdots + m\,s_m = \tfrac{1}{24}m^2(m^2{-}1)(m{+}2) = \dim K_1\big(\mathfrak{so}(m)\big),$$

as this latter number is the number of independent R'_{ijklq} that show up in the formulae for the derivatives of the R_{ijkl}:

$$\mathrm{d}R_{ijkl} = -R_{qjkl}\theta_{qi} - R_{iqkl}\theta_{qj} - R_{ijql}\theta_{qk} - R_{ijkq}\theta_{ql} + R'_{ijklq}\,\eta_q\,,$$

which are subject to the classical second Bianchi identity $R'_{ijklq}+R'_{ijqkl}+R'_{ijlqk} = 0$.

Thus, as expected, $A_{\mathfrak{so}(m)}$ is involutive, and the Riemannian metrics in dimension m (up to diffeomorphism) depend on $s_m = \tfrac{1}{2}m(m{-}1)$ functions of m variables. The above characters then determine the number of independent covariant derivatives of the curvature functions to any given order of differentiation.

Example 5 (Ricci-flat Kähler surfaces). When $H = \mathrm{SU}(2) \subset \mathrm{GL}(4,\mathbb{R})$, one is, in effect, considering Riemannian 4-manifolds with holonomy contained in $\mathrm{SU}(2)$. In this case, one finds that $\dim K_0(\mathfrak{h}) = 5$ and that the representation ρ_0 of $\mathrm{SU}(2)$ is irreducible. Indeed, one finds that the structure equations take the form

$$\begin{pmatrix} \mathrm{d}\eta_0 \\ \mathrm{d}\eta_1 \\ \mathrm{d}\eta_2 \\ \mathrm{d}\eta_3 \end{pmatrix} = - \begin{pmatrix} 0 & \theta_1 & \theta_2 & \theta_3 \\ -\theta_1 & 0 & -\theta_3 & \theta_2 \\ -\theta_2 & \theta_3 & 0 & -\theta_1 \\ -\theta_3 & -\theta_2 & \theta_1 & 0 \end{pmatrix} \wedge \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$$

and

$$\begin{pmatrix} \mathrm{d}\theta_1 \\ \mathrm{d}\theta_2 \\ \mathrm{d}\theta_3 \end{pmatrix} = - \begin{pmatrix} 2\,\theta_2{\wedge}\theta_3 \\ 2\,\theta_3{\wedge}\theta_1 \\ 2\,\theta_1{\wedge}\theta_2 \end{pmatrix} + \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} \eta_0{\wedge}\eta_1 - \eta_2{\wedge}\eta_3 \\ \eta_0{\wedge}\eta_2 - \eta_3{\wedge}\eta_1 \\ \eta_0{\wedge}\eta_3 - \eta_1{\wedge}\eta_2 \end{pmatrix},$$

where $R_{ij} = R_{ji}$ and $R_{11} + R_{22} + R_{33} = 0$.

It has already been shown that this defines a Jacobi manifold in $V \otimes \Lambda^2(V^*)$ where $V = \mathbb{R}^4 \oplus \mathfrak{su}(2) \simeq \mathbb{R}^7$, and its involutivity follows by inspection, since the characters are visibly $s_2 = 3$, $s_3 = 2$, and $s_k = 0$ all other k, and since the dimension of $K_1(\mathfrak{h})$ is easily computed to be $12 = 2s_2 + 3s_3$.

Thus, Theorem 4 applies and justifies Cartan's famous assertion that metrics in dimension 4 with holonomy $\mathrm{SU}(2)$ depend on $s_3 = 2$ arbitrary functions of three variables up to diffeomorphism.[15]

Example 6 (Segre structures of dimension $2m$). One can also apply these theorems to the study of 'higher order' H-structures, i.e., structures for which there is no canonical connection until after a prolongation has been performed.

Consider the generality of torsion-free $\mathrm{GL}(2,\mathbb{R})\cdot\mathrm{GL}(m,\mathbb{R})$-structures on $\mathbb{R}^{2m}$. In this discussion, I'm going to assume that $m > 2$, since the case $m = 2$ is equivalent to conformal structures of type $(2,2)$ on $\mathbb{R}^4$, which (as I'll point out below) turns out to have a different set of structure equations.

If $F \to U \subset \mathbb{R}^{2m}$ is a torsion-free $\mathrm{GL}(2,\mathbb{R})\cdot\mathrm{GL}(m,\mathbb{R})$-structure on $U \subset \mathbb{R}^{2m}$, then there is a prolongation of F to a second-order structure $F^{(1)}$, with structure group a semi-direct product of $\mathrm{GL}(2,\mathbb{R})\cdot\mathrm{GL}(m,\mathbb{R})$ with $\mathbb{R}^{2m}$, on which there exists a Cartan connection θ with values in $\mathrm{SL}(m{+}2,\mathbb{R})$, say

$$\theta = \begin{pmatrix} \psi^i_j & \eta^i_\beta \\ \omega^\alpha_j & \phi^\alpha_\beta \end{pmatrix},$$

where the index ranges are understood to be $1 \le i, j, k \le 2$ and $1 \le \alpha, \beta, \gamma \le m$, and the forms that are entries of θ satisfy the single trace relation $\psi^i_i + \phi^\alpha_\alpha = 0$ but are otherwise linearly independent. These components are required to satisfy structure equations of the form[16]

$$d\omega^\alpha_j = -\phi^\alpha_\beta \wedge \omega^\beta_j - \omega^\alpha_i \wedge \psi^i_j$$

$$d\psi^i_j = -\psi^i_k \wedge \psi^k_j - \eta^i_\beta \wedge \omega^\beta_j$$

$$d\phi^\alpha_\beta = -\phi^\alpha_\gamma \wedge \phi^\gamma_\beta - \omega^\alpha_i \wedge \eta^i_\beta + F^\alpha_{\beta\gamma\delta}\,\omega^\gamma_1 \wedge \omega^\delta_2$$

$$d\eta^i_\beta = -\psi^i_j \wedge \eta^j_\beta - \eta^i_\alpha \wedge \phi^\alpha_\beta + G^i_{\beta\gamma\delta}\,\omega^\gamma_1 \wedge \omega^\delta_2$$

$$dF^\alpha_{\beta\gamma\delta} = -F^\epsilon_{\beta\gamma\delta}\,\phi^\alpha_\epsilon + F^\alpha_{\epsilon\gamma\delta}\,\phi^\epsilon_\beta + F^\alpha_{\beta\epsilon\delta}\,\phi^\epsilon_\gamma + F^\alpha_{\beta\gamma\epsilon}\,\phi^\epsilon_\delta + R^{\alpha i}_{\beta\gamma\delta\epsilon}\,\omega^\epsilon_i$$

$$dG^i_{\beta\gamma\delta} = -G^j_{\beta\gamma\delta}\,\psi^i_j + G^i_{\epsilon\gamma\delta}\,\phi^\epsilon_\beta + G^i_{\beta\epsilon\delta}\,\phi^\epsilon_\gamma + G^i_{\beta\gamma\epsilon}\,\phi^\epsilon_\delta - F^\alpha_{\beta\gamma\delta}\,\eta^i_\alpha + Q^{ij}_{\beta\gamma\delta\epsilon}\,\omega^\epsilon_j.$$

The functions F, G, R, and Q must satisfy the relations

$$F^\alpha_{\beta\gamma\delta} = F^\alpha_{\gamma\beta\delta} = F^\alpha_{\beta\delta\gamma}, \qquad\qquad F^\alpha_{\alpha\gamma\delta} = 0,$$

$$G^i_{\beta\gamma\delta} = G^i_{\gamma\beta\delta} = G^i_{\beta\delta\gamma}$$

as well as the relations

$$R^{\alpha i}_{\beta\gamma\delta\epsilon} = P^{\alpha i}_{\beta\gamma\delta\epsilon} + \frac{1}{m{+}3}\left(\delta^\alpha_\beta\,G^i_{\gamma\delta\epsilon} + \delta^\alpha_\gamma\,G^i_{\beta\delta\epsilon} + \delta^\alpha_\delta\,G^i_{\beta\gamma\epsilon} - (m{+}2)\delta^\alpha_\epsilon\,G^i_{\beta\gamma\delta}\right),$$

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[15]"Les espaces de Riemann précédents dépendent de deux fonctions arbitraires de trois arguments..." ([6], pp. 55–56). As far as I know, Cartan never gave any justification for this assertion, which is the earliest case I know of in which an irreducible holonomy group is discussed, other than the case of symmetric spaces. It seems highly likely to me, though, that he was already, at that time (1926), aware of some version of Theorem 4.

[16]Here is where the assumption that $m > 2$ is important. The correct structure equations for $m = 2$ have nontrivial curvature terms in the structure equations for $d\psi^i_j$, as the reader can easily check. In fact, for $m = 2$, the structure equations as I have written them are the structure equations for the so-called 'half-flat' conformal structures of type $(2,2)$, i.e., the ones for which the self-dual part of the Weyl curvature vanishes.

where $P^{\alpha i}_{\beta\gamma\delta\epsilon}$ is fully symmetric in its lower indices and satisfies $P^{\alpha i}_{\alpha\beta\gamma\delta} = 0$. Finally, $Q^{ij}_{\beta\gamma\delta\epsilon}$ must be fully symmetric in its lower indices.

Note that, in the application of Theorem 4, the 1-forms play the role of the ω^i, the independent coefficients in F and G play the role of coordinates on the appropriate Jacobi manifold A, while the independent coefficients in P and Q play the role of coordinates on $A^{(1)}$.

While the number n is actually $(m+2)^2 - 1 = m^2 + 4m + 3$, it's also clear from the structure equations that only the ω^α_i are effectively involved in the computation of the characters (since it is only these terms that appear with non-constant coefficients in the structure equations). Thus (modulo what should be thought of as 'Cauchy characteristics'), the 'effective dimension' is $n = 2m$.

As the reader can check, the formal $\mathrm{d}^2 = 0$ conditions needed for Theorem 4 are satisfied. Using the symmetries of the coefficients, the dimensions

$$\dim A = (m+2)\binom{m+2}{3} - \binom{m+1}{2} = \frac{1}{6}\,m(m+1)(m^2+4m+1)$$

and

$$\dim A^{(1)} = (2m+4)\binom{m+3}{4} - 2\binom{m+2}{3} = \frac{1}{12}\,m(m+1)(m+2)(m^2+5m+2)$$

are easily computed.

It remains to compute the characters, which turn out to be

$$s_k = (k-1)\big(m^2 - (k-4)m - 2k + 3\big)$$

for $1 \leq k \leq m+1$ and $s_k = 0$ for $k > m+1$. Thus, A is an involutive Jacobi manifold.

In particular, up to diffeomorphism, the general such torsion-free structure depends on $s_{m+1} = m(m+1)$ functions of $m+1$ variables, and there exists such a structure taking any given desired curvature value.

Remark 15 (Torsion-free H-structures). For many other examples of this kind, examining the generality up to diffeomorphism of local torsion-free H-structures for various groups $H \subset \mathrm{GL}(m,\mathbb{R})$, the reader might consult [2] and [3]. Essentially all questions about the existence and generality of local torsion-free structures of this kind can be resolved by an application of Theorem 4.

5.6.2. *Other prescribed curvature conditions.* Sometimes one wants to consider a proper submanifold of $A_\mathfrak{h}$ in order to investigate H-structures with some extra condition on the curvature that captures some geometric property.

Example 7 (Einstein-Weyl structures). Consider Cartan's analysis [8] of the so-called *Einstein-Weyl structures* on 3-manifolds. These are $\mathrm{CO}(3)$-structures on 3-manifolds endowed with a compatible torsion-free connection whose curvature function takes values in a certain 4-dimensional submanifold $W \subset A_{\mathfrak{co}(3)}$.

Here are their structure equations as Cartan writes them (with a very slight change in notation):

$$\begin{pmatrix} \mathrm{d}\eta_1 \\ \mathrm{d}\eta_2 \\ \mathrm{d}\eta_3 \end{pmatrix} = - \begin{pmatrix} \theta_0 & \theta_3 & -\theta_2 \\ -\theta_3 & \theta_0 & \theta_1 \\ \theta_2 & -\theta_1 & \theta_0 \end{pmatrix} \wedge \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$$

and

$$\begin{pmatrix} d\theta_0 \\ d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \theta_2 \wedge \theta_3 \\ \theta_3 \wedge \theta_1 \\ \theta_1 \wedge \theta_2 \end{pmatrix} + \begin{pmatrix} 2H_1 & 2H_2 & 2H_3 \\ H_0 & H_3 & -H_2 \\ -H_3 & H_0 & H_1 \\ H_2 & -H_1 & H_0 \end{pmatrix} \begin{pmatrix} \eta_2 \wedge \eta_3 \\ \eta_3 \wedge \eta_1 \\ \eta_1 \wedge \eta_2 \end{pmatrix},$$

where the functions H_0, H_1, H_2, and H_3 are coordinates on W. This is a set of structure equations of the type to which Theorem 4 might apply, where the affine subspace $W \subset V \otimes \Lambda^2(V^*)$ has dimension 4 and where $V = \mathbb{R}^3 \oplus \mathbb{R} \oplus \mathfrak{so}(3) \simeq \mathbb{R}^7$. It is easy to verify that W is a Jacobi manifold and is involutive with $s_2 = 4$ and all other $s_k = 0$. Thus, Theorem 4 applies, and one recovers Cartan's result that the general Einstein-Weyl space depends on four arbitrary functions of two variables.

When $A_{\mathfrak{h}}$ is not involutive, one can ask whether its prolongation, which is got by adjoining the equation

(5.20) $$\mathrm{d}R = -\rho'_0(\theta)R + R'(\eta)$$

to the pair (5.18) and (5.19), is involutive, where R' takes values in the subspace $K_1(\mathfrak{h}) \subset K_0(\mathfrak{h}) \otimes \mathfrak{m}^*$ that is the kernel of the natural mapping

$$K_0(\mathfrak{h}) \otimes \mathfrak{m}^* \subset \mathfrak{h} \otimes \Lambda^2(\mathfrak{m}^*) \otimes \mathfrak{m}^* \to \mathfrak{h} \otimes \Lambda^3(\mathfrak{m}^*).$$

The combined system of equations (5.18), (5.19), and (5.20) is of the type that Theorem 3 was intended to treat, with R playing the role of the a^α and R' playing the role of the b^σ.

It may be necessary to repeat this prolongation process several times in order to arrive at a system of structure equations to which Theorem 3 can be applied.

Example 8 (Bochner-Kähler metrics). An interesting example is when $\mathfrak{m} = \mathbb{C}^n$ and $H = \mathrm{U}(n) \subset \mathrm{GL}(\mathfrak{m})$. In this case, one finds that $K_0(\mathfrak{h})$ is decomposable as a $\mathrm{U}(n)$-module into three irreducible summands,

$$K_0(\mathfrak{h}) = S(\mathfrak{h}) \oplus \mathrm{Ric}_0(\mathfrak{h}) \oplus B(\mathfrak{h}),$$

where $S(\mathfrak{h}) \simeq \mathbb{R}$ corresponds to the space of curvature tensors of Kähler manifolds with constant holomorphic sectional curvature, $\mathrm{Ric}_0(\mathfrak{h})$ corresponds to the space of traceless Ricci curvatures of Kähler metrics, and $B(\mathfrak{h})$, known as the space of *Bochner curvatures*, corresponds to the space of curvature tensors of Ricci-flat Kähler manifolds. A Kähler metric is said to be *Bochner-Kähler* if the $B(\mathfrak{h})$-component of its curvature tensor vanishes, i.e., if its curvature tensor takes values in $S(\mathfrak{h}) \oplus \mathrm{Ric}_0(\mathfrak{h})$.

Thus, the Bochner-Kähler condition defines a Jacobi manifold $A \subset K_0(\mathfrak{h})$ that turns out not to be involutive. However, after a succession of applications of the prolongation process (in fact, three prolongations), one arrives at a set of structure equations that has no free derivatives but satisfies the hypotheses of Theorem 2, thus showing that germs of Bochner-Kähler metrics depend on a finite number of constants. For details, see [4].

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