LECTURE 1: SOME THEOREMS OF LIE AND CARTAN

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Abstract. These are my notes for Eilenberg Lecture 1. The goal of this lecture is to introduce a generalization by Élie Cartan of Lie's Third Theorem (on the existence of a (local) Lie group associated to every Lie algebra) and then to illustrate the use of Cartan's generalization in a couple of low dimensional differential geometry problems.

The underlying theme is that the compatability assumptions needed to prove the existence of solutions various problems in differential geometry can be encoded as the condition that the differential forms of various degree that are involved generate a differential algebra, i.e., an algebra of differential forms endowed with a derivation d of degree +1 that satisfies $\mathrm{d}^2 = 0$.

1. Lie's Third Theorem and Cartan's Generalization

1.1. **Lie's Theorem.** In its original form, Lie's Third Fundamental Theorem can be stated as follows:

Theorem 1 (S. Lie, 1893?)**.** *Suppose that $C^i_{jk} = -C^i_{kj} \in \mathbb{R}$ (with $1 \le i, j, k \le n$) satisfy*

$$C^i_{mj}C^m_{kl} + C^i_{mk}C^m_{lj} + C^i_{ml}C^m_{jk} = 0. \qquad (\textit{the } \text{Jacobi condition}) \tag{1.1}$$

(1) *Existence: There exist linearly independent vector fields $X_1, \ldots, X_n$ on $\mathbb{R}^n$ satisfying*
$$[X_j, X_k] = C^i_{jk}\, X_i\,.$$
(2) *Uniqueness: For linearly independent $Y_1, \ldots, Y_n$ on $\mathbb{R}^n$ satisfying $[Y_j, Y_k] = C^i_{jk}\, Y_i$ and any $x_0, y_0 \in \mathbb{R}^n$, there exist open neighborhoods $U \ni x_0$ and $V \ni y_0$ and a diffeomorphism $\phi : U \to V$ satisfying $\phi(x_0) = y_0$ and $\phi_*(X_i) = Y_i$ (i.e., $\mathrm{d}\phi_x\big(X_i(x)\big) = Y_i\big(\phi(x)\big)$ for all $x \in U$.*

This result can be stated in dual form as follows

Theorem 2. *If $C^i_{jk} = -C^i_{kj}$ satisfy* (1.1)*, then there exist linearly independent* 1*-forms $\omega^1, \ldots, \omega^n$ on $\mathbb{R}^n$ satisfying*

$$\mathrm{d}\omega^i = -\tfrac{1}{2}\, C^i_{jk}\, \omega^j \wedge \omega^k. \tag{1.2}$$

Conversely, if (1.2) *holds for some linearly independent* 1*-forms $\omega^1, \ldots, \omega^n$ on $\mathbb{R}^n$, then* (1.1) *holds.*

Note that this last statement follows from (1.2) (and, in fact, is equivalent to it) by expanding out the consequence of

$$0 = \mathrm{d}\big(\mathrm{d}\omega^i\big) = -\tfrac{1}{2}\,\mathrm{d}\left(C^i_{jk}\, \omega^j \wedge \omega^k\right).$$

Of course, we now know a stronger, 'global' result:

Theorem 3 (Cartan-Lie)**.** *For any real constants $C^i_{jk} = -C^i_{kj}$ satisfying* (1.1)*, there exists a connected and simply-connected Lie group G of dimension n and a basis for its left-invariant* 1*-forms $\omega^1, \ldots, \omega^n$ satisfying* (1.2)*. Moreover, such G is unique up to Lie group isomorphism.*

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1.2. **Cartan's Generalization.** In 1904, Élie Cartan proved a generalization that can be regarded as a sort of 'parametrized version' of Lie's Theorem. He considered n-linearly independent 1-forms $\omega^1, \ldots, \omega^n$ on $\mathbb{R}^n$ that satisfy equations

$$\mathrm{d}\omega^i = -\tfrac{1}{2}\, C^i_{jk}(x^1, \ldots, x^s)\, \omega^j \wedge \omega^k, \tag{1.3}$$

where the $C^i_{jk} = -C^i_{jk}$ are no longer constant but are *given* functions of s other functions $x^1, \ldots, x^s$ on $\mathbb{R}^n$ that satisfy

$$\mathrm{d}x^a = F^a_i(x^1, \ldots, x^s)\, \omega^i \tag{1.4}$$

for some *given* functions F^a_i on $\mathbb{R}^s$.

Given such a situation, the identity $\mathrm{d}(\mathrm{d}x^a) = 0$ implies that the equations

$$F^b_i \frac{\partial F^a_j}{\partial x^b} - F^b_j \frac{\partial F^a_i}{\partial x^b} = F^a_l C^l_{ij}\,,$$

hold on $\mathbb{R}^n$ while the identity $\mathrm{d}(\mathrm{d}\omega^i) = 0$ implies that the equations

$$F^a_j \frac{\partial C^i_{kl}}{\partial x^a} + F^a_k \frac{\partial C^i_{lj}}{\partial x^a} + F^a_l \frac{\partial C^i_{jk}}{\partial x^a} = \left(C^i_{mj} C^m_{kl} + C^i_{mk} C^m_{lj} + C^i_{ml} C^m_{jk} \right).$$

hold on $\mathbb{R}^n$

In [2], Cartan proved a sort of converse existence theorem

Theorem 4 (Cartan)**.** *Suppose that $A \subset \mathbb{R}^s$ is an open set and suppose that F^a_i and $C^i_{jk} = -C^i_{kj}$ for $1 \le a, b \le s$ and $1 \le i, j, k \le n$ are real-analytic functions on A that satisfy*

$$F^b_i \frac{\partial F^a_j}{\partial x^b} - F^b_j \frac{\partial F^a_i}{\partial x^b} = F^a_l C^l_{ij} \tag{1.5}$$

and

$$F^a_j \frac{\partial C^i_{kl}}{\partial x^a} + F^a_k \frac{\partial C^i_{lj}}{\partial x^a} + F^a_l \frac{\partial C^i_{jk}}{\partial x^a} = \left(C^i_{mj} C^m_{kl} + C^i_{mk} C^m_{lj} + C^i_{ml} C^m_{jk} \right) \tag{1.6}$$

on X.

Then for every $h_0 \in A$, there exists a real-analytic n-manifold N with a real-analytic coframing $\eta = (\eta^i)$ and a real-analytic function $h : N \to A$ satisfying the structure equations

$$d\eta^i = -\tfrac{1}{2} C^i_{jk}(h)\, \eta^j \wedge \eta^k, \qquad dh^a = F^a_i(h)\, \eta^i\,. \tag{1.7}$$

and a $p_0 \in N$ for which $h(p_0) = h_0$.

Moreover, this 'realization' (N, η, h) is locally unique in the following sense: Given any other real-analytic realization $(\tilde{N}, \tilde{\eta}, \tilde{h})$ satisfying the corresponding structure equations that contains a point $\tilde{p}_0 \in \tilde{N}$ satisfying $\tilde{h}(\tilde{p}_0) = h_0$, there exists a p_0-neighborhood $U \subset N$, a $\tilde{p}_0$-neighborhood $\tilde{U} \subset \tilde{N}$, and a real-analytic diffeomorphism $\phi : \tilde{U} \to U$ so that

$$\phi(\tilde{p}_0) = p_0, \qquad \phi^*(\eta) = \tilde{\eta}, \qquad \textit{and} \;\; \phi^*(h) = \tilde{h}.$$

Essentially, this theorem asserts that, if the structure equations (1.7) satisfy $\mathrm{d}^2 = 0$ 'formally', then there exist 'realizations' (η, h) of these structure equations with the invariant function h taking any value in A. Moreover, locally, up to diffeomorphism, such a realization is determined by the value h_0 of its 'invariants' h at a single point.

Note that Lie's original theorem concerns the special case in which $s = 0$, i.e., the 'functions' C^i_{jk} are constants.

Remark 1 (Smoothness and Localness). Cartan did not explicitly assume real-analyticity in his statement, but, in fact, his proof uses a PDE existence theorem that is only valid in the real-analytic category. Later developments in the 1970s allowed the removal of the assumption of real-analyticity, but at the expense of requiring a considerably more delicate proof.

Also, it turns out that, unlike the case of Lie's theorem, there is no corresponding 'global' statement, even when we generalize the open set $A \subset \mathbb{R}^s$ to a smooth s-manifold. (This 'globalization' has been done, and the structure implicit in the statement of Theorem 4, nowadays goes under the name *Lie algebroid*. I will say more about this later.)

2. Some geometric examples

One can reasonably ask what Cartan's generalization is good for. It turns out that it's a convenient way to prove (local) existence of solutions of various geometric problems involving PDE. I'll give a couple of examples here to illustrate the possibilities, but we'll see many more serious examples later on.

2.1. **Hessian equations for surfaces.** A Riemannian metric g on a manifold M^n allows one to define the *Hessian* of a function f on M, which is the quadratic form $\mathrm{Hess}_g(f) = \nabla(\mathrm{d}f)$.

For a Riemannian surface (M^2, g), we have its Gauss curvature K and one can investigate conditions on $\mathrm{Hess}(K) = \nabla(\mathrm{d}K)$. A natural condition would be to ask when $\mathrm{Hess}(K)$ can be expressed in terms of lower order expressions in K. I will consider equations of the form

$$\mathrm{Hess}(K) = a(K)\, g + b(K)\, (\mathrm{d}f)^2, \tag{2.1}$$

where a and b are specified functions of one variable. This equation is, of course, diffeomorphism invariant, so the goal would be to ask what one can say about such surfaces up to diffeomorphism.

Writing $g = {\omega_1}^2 + {\omega_2}^2$ on the orthonormal frame bundle F^3 of M, the structure equations become

$$\begin{aligned} \mathrm{d}\omega_1 &= -\omega_{12} \wedge \omega_2 & \qquad \mathrm{d}\omega_{12} &= K\, \omega_1 \wedge \omega_2 \\ \mathrm{d}\omega_2 &= \omega_{12} \wedge \omega_1 & \mathrm{d}K &= K_1\, \omega_1 + K_2\, \omega_2\,, \end{aligned} \tag{2.2}$$

where ω_{12} is the connection form on F that defines the Levi-Civita connection of g on M. Note that the relations $\mathrm{d}(\mathrm{d}\omega_1) = \mathrm{d}(\mathrm{d}\omega_2) = \mathrm{d}(\mathrm{d}\omega_1) = 0$ are formally implied by the above structure equations, but $\mathrm{d}(\mathrm{d}K)$ does not automatically vanish without making assumptions about $\mathrm{d}K_1$ and $\mathrm{d}K_2$. In fact, $\mathrm{d}(\mathrm{d}K) = 0$ implies that there must exist functions $K_{ij} = K_{ji}$ so that

$$\begin{pmatrix} \mathrm{d}K_1 \\ \mathrm{d}K_2 \end{pmatrix} = \begin{pmatrix} -K_2 \\ K_1 \end{pmatrix} \omega_{12} + \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix},$$

and, in this case, we have $\nabla(\mathrm{d}K) = K_{ij}\, \omega_i \otimes \omega_j = K_{ij}\, \omega_i \circ \omega_j$.

Since $a(K)g + b(K)\, (\mathrm{d}K)^2$ expands to

$$\big(a(K) + b(K)\, {K_1}^2\big)\, \omega_1 \circ \omega_1 + 2b(K)\, K_1 K_2\, \omega_1 \circ \omega_2 + \big(a(K) + b(K)\, {K_2}^2\big)\, \omega_2 \circ \omega_2\,,$$

the condition to be studied is encoded as

$$\begin{pmatrix} \mathrm{d}K_1 \\ \mathrm{d}K_2 \end{pmatrix} = \begin{pmatrix} -K_2 \\ K_1 \end{pmatrix} \omega_{12} + \begin{pmatrix} a(K) + b(K)\, {K_1}^2 & b(K)\, K_1 K_2 \\ b(K)\, K_1 K_2 & a(K) + b(K)\, {K_2}^2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}. \tag{2.3}$$

Now $\mathrm{d}(\mathrm{d}K) = 0$ is an identity, and applying $\mathrm{d}^2 = 0$ to the last two equations yields the relations

$$\big(a'(K) - a(K)b(K) + K\big)\, K_i = 0 \qquad \text{for } i = 1, 2.$$

Thus, unless $a'(K) = a(K)b(K) - K$, such metrics must have K constant. The original equation can then only be satisfied if $a \equiv 0$. For example, there are no metrics (even locally) that satisfy $\mathrm{Hess}(K) = g$ (where $a(K) \equiv 1$ and $b(K) \equiv 0$).

On the other hand, suppose that the functions a and b satisfy

$$a'(K) = a(K)b(K) - K.$$

Does there exist a 'solution' $(\omega_1, \omega_2, \omega_{12}, K, K_1, K_2)$ to the following system?

$$\begin{aligned} \mathrm{d}\omega_1 &= -\omega_{12}\wedge\omega_2 \\ \mathrm{d}\omega_2 &= \quad\omega_{12}\wedge\omega_1 \qquad\qquad \omega_1\wedge\omega_2\wedge\omega_{12}\neq 0, \\ \mathrm{d}\omega_{12} &= K\,\omega_1\wedge\omega_2 \end{aligned}$$

where

$$\begin{pmatrix}\mathrm{d}K\\ \mathrm{d}K_1\\ \mathrm{d}K_2\end{pmatrix} = \begin{pmatrix} K_1 & K_2 & 0\\ a(K)+b(K)\,{K_1}^2 & b(K)\,K_1K_2 & -K_2\\ b(K)\,K_1K_2 & a(K)+b(K)\,{K_2}^2 & K_1\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_{12}\end{pmatrix}.$$

Since $\mathrm{d}^2 = 0$ is formally satisfied for these structure equations, Theorem 4, applies. Thus, for example, there are, at least locally, metrics that satisfy $\mathrm{Hess}(K) = g + K\,(\mathrm{d}K)^2$ (i.e., $a(K)\equiv 1$ and $b(K)\equiv K$). Moreover, one can specify values of K and $\mathrm{d}K$ at one point on the surface, and that will determine a metric satisfying (2.1), up to local diffeomorphism in a neighborhood of that point.

Global existence is another question, which we will take up later.

2.2. **Third order equations for Riemannian 3-manifolds.** For a Riemannian 3-manifold (M^3, g), the curvature can be expressed in terms of the Ricci curvature $\mathrm{Ric}(g)$, which is a symmetric quadratic tensor that depends on two derivatives of the metric g.

Because of the second Bianchi identity, the covariant derivative $\nabla\big(\mathrm{Ric}(g)\big)$, is a tensor of rank 15, which turns out to be the sum of three irreducible tensors of ranks 3, 5, and 7. Symbolically,

$$\nabla\big(\mathrm{Ric}(g)\big) = A_3 + B_5 + C_7.$$

The tensor A_3 is, up to a constant multiple, the exterior derivative of the scalar curvature of g. The tensor B_5, a traceless symmetric quadratic form, is the Cotton tensor, which vanishes if and only if the metric g is conformally flat. The tensor C_7, which does not appear to have an name, is a traceless symmetric cubic form.

While the metrics with $A_3 = 0$ are the constant scalar curvature metrics and the metrics with $B_5 = 0$ are the conformally flat metrics, for a long time there was no known characterization or description of metrics with $C_7 = 0$.

In 2019, Ryan Gunderson [3] proved that a 3-manifold with $C_7 = 0$ is either conformally flat (and so also satisfies $B_5 = 0$) or has an abelian isometry group of dimension 2. He was also able to characterize the solutions of each kind. His proof is complicated, requiring sophisticated use of symbolic computations, and I will say more about this later, when we discuss the use of such software, but, today, I want to treat a special case as an illustration of the use of Cartan's theorem.

The case I want to consider is the system $B_5 = C_7 = 0$.

To see how this works, first consider the structure equations for a Riemannian metric g on a 3-manifold M: Let $\pi : F \to M$ be the O(3)-bundle of g-orthonormal coframes. Thus, an element of $\pi^{-1}(x) = F_x$ is a linear isometry $u : T_xM \to \mathbb{R}^3$ where, for $x \in M$, T_xM is given its inner product g_x. One has a canonical $\mathbb{R}^3$-valued 1-form $\omega = (\omega_i) : TF \to \mathbb{R}^3$ defined by $\omega_u = \pi'(u)\circ u : T_uF \to \mathbb{R}^3$. Then

$$\pi^*(g) = \omega\cdot\omega = \omega^T\circ\omega = {\omega_1}^2 + {\omega_2}^2 + {\omega_3}^2.$$

The Levi-Civita connection form is the $\mathfrak{so}(3)$-valued 1-form $\theta = -\theta^T$ that satisfies the first structure equation

$$\mathrm{d}\omega = -\theta\wedge\omega. \tag{2.4}$$

Using Cartan's famous 'omitted index' notation, in which $\theta_{12} = \theta_3$, $\theta_{31} = \theta_2$, etc., we can write

$$\theta = \begin{pmatrix} 0 & \theta_3 & -\theta_2\\ -\theta_3 & 0 & \theta_1\\ \theta_2 & -\theta_1 & 0\end{pmatrix}.$$

Then the second structure equation can be written in the form

$$\begin{pmatrix} \mathrm{d}\theta_1 \\ \mathrm{d}\theta_2 \\ \mathrm{d}\theta_3 \end{pmatrix} = \begin{pmatrix} \theta_2{\wedge}\theta_3 \\ \theta_3{\wedge}\theta_1 \\ \theta_2{\wedge}\theta_3 \end{pmatrix} + K \begin{pmatrix} \omega_2{\wedge}\omega_3 \\ \omega_3{\wedge}\omega_1 \\ \omega_2{\wedge}\omega_3 \end{pmatrix}. \tag{2.5}$$

with

$$K = K^T = \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{pmatrix}$$

In more standard notation, $K_{11} = R_{2323}$, $K_{12} = R_{2331}$, etc. The second Bianchi identity, obtained by differentiating the second structure equation, can be written as

$$0 = DK \wedge \begin{pmatrix} \omega_2{\wedge}\omega_3 \\ \omega_3{\wedge}\omega_1 \\ \omega_2{\wedge}\omega_3 \end{pmatrix},$$

where

$$DK = \mathrm{d}K - K\,\theta + \theta\,K = (DK_{ij}) = \big(K_{ijk}\,\omega_k\big),$$

with $K_{ijk} = K_{jik}$, and this is equivalent to the three equations

$$K_{ijj} = 0.$$

It is not hard to show that one can write K_{ijk} uniquely in the form

$$K_{ijk} = \big(4\delta_{ij}\,a_k - \delta_{ik}\,a_j - \delta_{jk}\,a_i\big) + \big(\varepsilon_{ikl}\,b_{jl} + \varepsilon_{jkl}\,b_{il}\big) + c_{ijk}$$

where $b_{ij} = b_{ji}$ and $b_{jj} = 0$, where $c_{ijk} = c_{jik} = c_{ikj}$ and $c_{ijj} = 0$, and where δ_{ij} is the Kronecker delta and ε_{ijk} is totally skew-symmetric with $\varepsilon_{123} = 1$. The tensors $A_3 = a_i\,\omega_i$, $B_5 = b_{ij}\,\omega_i \circ \omega_j$, and $C_7 = c_{ijk}\,\omega_i \circ \omega_j \circ \omega_k$ are then the irreducible components of the first covariant derivative of the curvature tensor.

Henceforth, let us assume that $B_5 = C_7 = 0$. The second Bianchi identity now takes the form

$$\mathrm{d}K = K\,\theta - \theta\,K + 4(a^T\omega)I_3 - a\,\omega^T - \omega\,a^T. \tag{2.6}$$

Now, the system of equations given by (2.4), (2.5), and (2.6), does not have the form to which Theorem 4 can be applied because we do not have formulae for the differentials of the a_i.

However, we can get information about the differentials of the a_i by taking the exterior derivative of (2.6), using the structure equations we have derived to far.

The result, turns out to be that $\mathrm{d}(\mathrm{d}K) = 0$ implies that there is a function s such that

$$\mathrm{d}a = -\theta\,a + (sI_3 + \mathrm{tr}(K)\,K - K^2)\,\omega \tag{2.7}$$

Finally expanding out $\mathrm{d}(\mathrm{d}a) = 0$ implies that s must satisfy

$$\mathrm{d}s = \big(-6\ \mathrm{tr}(K)\,a^T + 2\,a^T\,K\,\big)\omega, \tag{2.8}$$

and, given the structure equations derived so far, we find that $\mathrm{d}(\mathrm{d}s) = 0$.

Thus, the system of equations (2.4), (2.5), (2.6), (2.7), and (2.8) satisfy the conditions of Theorem 4: There are six linearly independent 1-forms (the components of ω and θ) and ten 0-forms (the components of K, a, and s). Moreover, the given equations imply $\mathrm{d}(\mathrm{d}f) = 0$ for $f \in \{\omega, \theta, K, a, s\}$.

Theorem 4 then implies that one can specify at one point the values of the tensors $\kappa = K_{ij}\,\omega_i \circ \omega_j$, $\alpha = a_i\,\omega_i$, and the function s, and there will be a metric g, unique up to (pointed) local diffeomorphism, taking those values at a point and satisfying the 12 third-order equations $B_5 = C_7 = 0$.

Of course, it is not clear whether such a metric is complete or what the global properties of such metrics might be. That's a topic for another lecture.

3. Closing thoughts

It turns out that Theorem 4 is very useful in many cases to help one determine whether or not some geometrically defined condition is satisfiable, particularly in the 'overdetermined' case, where one expects there to be at most a finite-dimensional 'moduli space' of solutions. Of course, it's not clear how one might develop such an expectation of 'finiteness'. This is one of the things that I want to talk about in the next few weeks.

For example, while $B_5 = C_7 = 0$ turned out to have only a finite-dimensional moduli space of local solutions, that was not at all clear at the outset. Even less clear is that $C_7 = 0$ also has a finite-dimensional 'moduli space' of local solutions and that the moduli space turns out to have two distinct 'irreducible' components. In general, how can one determine these things?

Most geometric problems don't have only a finite-dimensional moduli space of local solutions. Is there some analog of Theorem 4 in such cases? What kind of information does it give? We will see that there is a systematic way to answer such questions, which will lead to the Cartan-Kähler Existence Theorem and, ultimately, the Cartan-Kuranishi Prolongation Theorem.

In the next few lectures, I will motivate these ideas and develop the proof strategies. Then, for the remainder of this series, I will look at some interesting applications and discuss what kinds of questions the current state of the theory hopes to address.

References

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