LECTURE 2: SOME CLASSICAL PDE BACKGROUND AND GENERALIZATIONS

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Abstract. These are my notes for Eilenberg Lecture 2. The goal of this lecture is to recall the classical Cauchy-Kovalevskaya Theorem, as a first-order, non-degenerate determined system of PDE, describe the strategy for applying it to systems that are either degenerate in some way or even overdetermined, and pave the way for the Cartan-Kähler Theorem.

1. The Cauchy-Kovalevskaya Theorem

One of the most basic results in PDE is the following, proved by Sofya Kovalevskaya in 1874.

Theorem 1 (Cauchy-Kovalevskaya)**.** *Let $F : \mathbb{R}^{n+1+s+sn} \to \mathbb{R}^s$ be a real-analytic function and let $\phi : \mathbb{R}^n \to \mathbb{R}^s$ be a real-analytic mapping. Then there is an open neighborhood $D \subset \mathbb{R}^n \times \mathbb{R}$ of $\mathbb{R}^n \times \{0\}$ on which there exists a real-analytic function $u : D \to \mathbb{R}^s$ satisfying the partial differential equation with initial condition*

$$\frac{\partial u}{\partial t} = F\left(x, t, u, \frac{\partial u}{\partial x}\right), \qquad u(x, 0) = \phi(x). \tag{1.1}$$

Moreover, if $D' \subset \mathbb{R}^n \times \mathbb{R}$ is another open neighborhood of $\mathbb{R}^n \times \{0\}$ on which there exists a $u' : D' \to \mathbb{R}^s$ satisfying (1.1)*, then there is an open neighborhood $D'' \subset D \cap D'$ of $\mathbb{R}^n \times \{0\}$ on which u and u' are equal.*

Remark 1. It's clear that, if a real-analytic solution u to (1.1) exists, it is given by a power series in t that begins

$$u = \phi + F\left(x, 0, \phi, \frac{\partial \phi}{\partial x}\right) t + \cdots .$$

The main issue is to show that the formal series converges on some open neighborhood D of the hyperplane $t = 0$. This is done by the *method of majorants*, as initiated by Cauchy, but the specific detailed argument needed to apply this method to the convergence of the above series was provided by Kovalevskaya.

Simple examples show that the assumption of real-analyticity on F and ϕ is needed for both the existence of (real-analytic) solutions u and their (local) uniqueness. Real-analyticity can be replaced by other assumptions on F and ϕ to get other theorems about existence and uniqueness of solutions, but I won't discuss those right now.

2. Treating a PDE system as a sequence of Cauchy problems

There are many real-analytic PDE to which the Cauchy-Kovalevskaya Theorem does not apply directly, but it can still be used to solve initial value problems. Here are a few illustrations of what kinds of things can happen.

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2.1. **Effects of choice of independent variables.** Consider the PDE system for $u : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\frac{\partial u_1}{\partial x^1} = \frac{\partial u_2}{\partial x^2} = 0. \tag{2.1}$$

If we filter this by derivatives with respect to the independent variables in the obvious way, the 'nested sequence of Cauchy problems' would be

$$\frac{\partial u_1}{\partial x^1} = 0 \quad \text{with solution} \quad \bigl(u_1(x^1, 0), u_2(x^1, 0)\bigr) = \bigl(c_1, \phi(x^1)\bigr). \tag{2.2}$$

but then as the initial condition for

$$\frac{\partial u_2}{\partial x^2} = 0 \quad \text{with solution} \quad \bigl(u_1(x^1, x^2), u_2(x^1, x^2)\bigr) = \bigl(c_1 + x^2\psi(x^1, x^2), \phi(x^1)\bigr), \tag{2.3}$$

which does not satisfy the first equation unless $\psi(x^1, x^2)$ is independent of x^1.

Obviously, this is a problem because of the choice of independent variables. If we set $x^1 = y^1 + y^2$ and $x^2 = y^1 - x^2$, then the original system becomes

$$\frac{\partial u_1}{\partial y^1} - \frac{\partial u_1}{\partial y^2} = \frac{\partial u_2}{\partial y^1} + \frac{\partial u_2}{\partial y^2} = 0, \tag{2.4}$$

which can be written in Cauchy form

$$\begin{aligned} \frac{\partial u_1}{\partial y^2} &= \frac{\partial u_1}{\partial y^1} \\ \frac{\partial u_2}{\partial y^2} &= -\frac{\partial u_2}{\partial y^1}, \end{aligned}$$

so the initial value problem, specifying u_1 and u_2 along $y^2 = 0$ is (uniquely) solvable.

Clearly, some care has to be taken in 'filtering' even simple PDE systems if the strategy of writing a system as a sequence of Cauchy problems is going to be successful.

2.2. **A 'degenerate' equation.** For V a vector field on $\mathbb{R}^3$, and given $F : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, consider the determined equation curl $V = F(x, V)$. Setting $V = (u_1, u_2, u_3)$, this becomes

$$\begin{aligned} \frac{\partial u_3}{\partial x^2} - \frac{\partial u_2}{\partial x^3} &= F^1(x, u) \\ \frac{\partial u_1}{\partial x^3} - \frac{\partial u_3}{\partial x^1} &= F^2(x, u) \\ \frac{\partial u_2}{\partial x^1} - \frac{\partial u_1}{\partial x^2} &= F^3(x, u). \end{aligned} \tag{2.5}$$

This determined system of equations cannot be put in Cauchy-Kovalevskaya form, i.e., solved for the three $\partial u_i/\partial x^k$ for some specific $k = 1$, 2, or 3.

In fact, the above equations would imply an 'extra' first-order equation on the functions u_i:

$$\frac{\partial}{\partial x^1}\bigl(F^1(x, u)\bigr) + \frac{\partial}{\partial x^2}\bigl(F^2(x, u)\bigr) + \frac{\partial}{\partial x^3}\bigl(F^3(x, u)\bigr) = 0 \tag{2.6}$$

which need not have any solutions at all, for any initial conditions. For example, consider the specific case $F^i(x, u) = x^i$.

Even if not obviously 'incompatible', how could we know whether there are solutions, or describe the general solution?

Consider the specific case of $F = (F^1, F^2, F^3) = (u_1, u_2, u_3)$, i.e., the equation cur$(V) = V$, together with the 'extra' equation div$(V) = 0$. It's clear that you can't specify a solution V arbitrarily along $x^3 = 0$, say, because one of the equations is

$$\frac{\partial u_2}{\partial x^1} - \frac{\partial u_1}{\partial x^2} = u_3$$

so an initial condition that leads to a solution would have to satisfy

$$u_3(x^1,x^2,0) = \frac{\partial u_2}{\partial x^1}(x^1,x^2,0) - \frac{\partial u_1}{\partial x^2}(x^1,x^2,0).$$

We can extract a system in Cauchy-Kovalevskaya form from the four first-order equations. We have:

$$\begin{aligned}\frac{\partial u_1}{\partial x^3} &= \frac{\partial u_3}{\partial x^1} + u_2\\ \frac{\partial u_2}{\partial x^3} &= \frac{\partial u_3}{\partial x^2} - u_1\\ \frac{\partial u_3}{\partial x^3} &= -\frac{\partial u_1}{\partial x^1} - \frac{\partial u_2}{\partial x^2}\end{aligned}$$

and try to solve this with the initial condition

$$\big(u_1(x^1,x^2,0), u_2(x^1,x^2,0), u_3(x^1,x^2,0)\big) = \left(\phi_1(x^1,x^2), \phi_2(x^1,x^2), \frac{\partial \phi_2}{\partial x^1}(x^1,x^2) - \frac{\partial \phi_1}{\partial x^2}(x^1,x^2)\right).$$

If ϕ_1 and ϕ_2 are analytic functions on $\mathbb{R}^2$, the Cauchy-Kovalevskaya Theorem guarantees a local real-analytic solution u to this system with this initial condition.

However, how do we know that this satisfies the original system? Well, we have the 'magic' identity

$$\frac{\partial}{\partial x^3}\left(\frac{\partial u_2}{\partial x^1} - \frac{\partial u_1}{\partial x^2} - u_3\right) = \frac{\partial}{\partial x^1}\left(\frac{\partial u_2}{\partial x^3} - \frac{\partial u_3}{\partial x^2} + u_1\right) - \frac{\partial}{\partial x^2}\left(\frac{\partial u_1}{\partial x^3} - \frac{\partial u_3}{\partial x^1} - u_2\right) - \left(\frac{\partial u_1}{\partial x^1} + \frac{\partial u_2}{\partial x^2} + \frac{\partial u_3}{\partial x^3}\right),$$

Thus the expression

$$E = \frac{\partial u_2}{\partial x^1} - \frac{\partial u_1}{\partial x^2} - u_3$$

is independent of x^3. Since, by our choice of initial conditions, we have $E(x^1,x^2,0) = 0$, we see that $E(x^1,x^2,x^3) = 0$, so the solution of this Cauchy problem solves the original system.

In other words, our solution to the Cauchy-Kovalevskaya system is actually a solution to the original system, and this process, with a judicious choice of initial conditions (which allows us to choose two 'arbitrary' real-analytic functions of two variables), generates the 'general' solution.

The existence of the above identity may seem like a lucky 'accident', but, it turns out that this 'trick' of combining $\mathrm{curl}(V) = F(x,V)$ with the equation $\mathrm{div}\big(F(x,v)\big) = 0$ to make an overdetermined system that can be treated as a sequence of initial value problems works far more generally:

Theorem 2. *If F is analytic and $Q^{ij} = \left(\frac{\partial F^i}{\partial u_j} + \frac{\partial F^i}{\partial u_i}\right)$ is nonzero at $(\bar{x},\bar{u}) \in \mathbb{R}^6$, then there exists a solution $u = \phi(x)$ to the equation* curl $u = F(x,u)$ *with $\phi(\bar{x}) = \bar{u}$ on some open neighborhood of $\bar{x} \in \mathbb{R}^3$. Moreover, the 'general' local real-analytic solution depends on two arbitrary (analytic) functions of two variables.*

We'll see a proof of this result (and a more precise statement) after we discuss the Cartan-Kähler Theorem.

2.3. **Total systems.** A first order system of PDE for a mapping $u : \mathbb{R}^n \to \mathbb{R}^s$ that specifies *all* the partials of the components of u as given functions of x and u is said to be a *total system*, i.e., of the form

$$\frac{\partial u^a}{\partial x^i} = F^a_i(x,u).$$

It is not hard to see that this can be written as a 'nested' sequence of Cauchy problems that have local solutions for all initial conditions $u(x_0) = u_0$ if and only if the derived expressions for the mixed partials agree, i.e., if and only if the functions F^a_i on $\mathbb{R}^{n+s}$ satisfy the identities

$$\frac{\partial}{\partial x^i}\left(\frac{\partial u^a}{\partial x^j}\right) - \frac{\partial}{\partial x^j}\left(\frac{\partial u^a}{\partial x^i}\right) = \frac{\partial F^a_i}{\partial x^j} - \frac{\partial F^a_j}{\partial x^i} - F^b_i\frac{\partial F^a_j}{\partial u^b} + F^b_j\frac{\partial F^a_i}{\partial u^b} = 0.$$

Of course, this is exactly the condition that the vector fields

$$X_i = \frac{\partial}{\partial x^i} + F^a_i(x,u)\,\frac{\partial}{\partial u^a}$$

satisfy $[X_i, X_j] = 0$. (In 1872, S. Lie referred to vector fields X and Y satisfying $[X,Y] = 0$ as being *in involution* or *involutive.*)

Of course, this is equivalent to the condition that the exterior ideal generated by the s linearly independent one-forms on $\mathbb{R}^{n+s}$ defined by

$$\theta^a = \mathrm{d}u^a - F^a_i(x,u)\,\mathrm{d}x^i$$

be *differentially closed*, i.e., $\mathrm{d}\theta^a = \psi^a_b{\wedge}\theta^b$ for some 1-forms ψ^a_b. (In 1872, G. F. Frobenius proved a statement that mounted to the claim that *any* system of s linearly independent one-forms $\theta^1,\ldots,\theta^s$ that generate a differentially closed exterior ideal is locally generated by s *exact* differentials, i.e., of the form $\{\mathrm{d}z^1,\ldots,\mathrm{d}z^s\}$. Such systems came to be called *involutive.*)

2.4. **An example that needs further treatment.** Even when a system cannot directly be filtered into a sequence of Cauchy problems as above, it's sometimes possible to replace the system by an equivalent system for which this idea does work.

Consider the first-order system for $u : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\frac{\partial u_1}{\partial x^1} = \frac{\partial u_1}{\partial x^2} - \frac{\partial u_2}{\partial x^1} = \frac{\partial u_2}{\partial x^2} = 0.$$

this can be written as two systems:

$$\text{(I)}\quad \frac{\partial u_1}{\partial x^1} = 0, \qquad \text{with solution} \qquad \bigl(u_1(x^1,0), u_2(x^1,0)\bigr) = \bigl(a, \phi(x^1)\bigr),$$

where a is a constant, but then the system with initial condition

$$\text{(II)}\quad \frac{\partial u_1}{\partial x^2} - \frac{\partial u_2}{\partial x^1} = \frac{\partial u_2}{\partial x^2} = 0 \qquad \text{with initial condition} \qquad \bigl(u_1(x^1,0), u_2(x^1,0)\bigr) = \bigl(a, \phi(x^1)\bigr)$$

has the solution

$$\bigl(u_1(x^1,x^2), u_2(x^1,x^2)\bigr) = \bigl(a{+}\phi'(x^1)x^2, \phi(x^1)\bigr),$$

which does not satisfy the first system unless $\phi'' = 0$.

This is not just a problem caused by a bad choice of coordinates. As we will see, the failure lies deeper than that.

Nevertheless, it is possible to 'prolong' this system to a larger system that can be 'filtered' into a nested sequence of Cauchy problems: Consider the system for $u : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\frac{\partial u_1}{\partial x^1} = \frac{\partial u_1}{\partial x^2} - u_3 = \frac{\partial u_2}{\partial x^1} - u_3 = \frac{\partial u_2}{\partial x^2} = \frac{\partial u_3}{\partial x^1} = \frac{\partial u_3}{\partial x^2} = 0.$$

This can be filtered into the nested systems

$$\begin{aligned} \frac{\partial u_1}{\partial x^1} &= 0 \\ \frac{\partial u_2}{\partial x^1} &= u_3 \\ \frac{\partial u_3}{\partial x^1} &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \frac{\partial u_1}{\partial x^2} &= u_3 \\ \frac{\partial u_2}{\partial x^2} &= 0 \\ \frac{\partial u_3}{\partial x^2} &= 0 \end{aligned}$$

for which it is easy to see that any solution of the first system gives a set of initial conditions that, when used for the second system, generates a solution to both systems: The general solution of the first system gives the initial condition

$$\bigl(u_1(x^1,0), u_2(x^1,0), u_3(x^1,0),\bigr) = (c_1, c_2 + c_3\,x^1, c_3)$$

for some constants c_1, c_2, and c_3, and, using this as initial conditions for the second system yields the solution

$$\bigl(u_1(x^1,x^2),u_2(x^1,x^2),u_3(x^1,x^2),\bigr) = (c_1 + c_3\,x_2, c_2 + c_3\,x^1, c_3),$$

which, for the original unknowns u_1 and u_2, is the general solution of the original system. Of course, it's not immediately apparent why one should add the two PDE for u_3. This is something that we will take up next time.

2.5. **Isometric embedding of surfaces.** An interesting differential geometry problem that does not, immediately, fit into this 'nested Cauchy systems' scheme is the problem of isometric embedding: Given a Riemannian metric

$$g = E(x,y)\,\mathrm{d}x^2 + 2F(x,y)\,\mathrm{d}x\mathrm{d}y + G(x,y)\,\mathrm{d}y^2,$$

on a domain D in the xy-plane, one would like to show that there is an immersion $u = (u_1,u_2,u_3) : D \to \mathbb{R}^3$, that induces the metric g on D, i.e.

$$g = \mathrm{d}u\cdot\mathrm{d}u = u_x{\cdot}u_x\,\mathrm{d}x^2 + 2u_x{\cdot}u_y\,\mathrm{d}x\mathrm{d}y + u_y{\cdot}u_y\,\mathrm{d}y^2.$$

This is three first-order PDE for the three components of u, but it cannot, even locally be written in the form

$$\frac{\partial u}{\partial y} = F\left(x,y,\frac{\partial u}{\partial x}\right).$$

Nevertheless, as we will see, the solutions can be written as the solutions to a sequence of 'nested Cauchy problems with initial conditions'.

3. Cartan's Reformulation and Theorem

Élie Cartan developed a geometric way to apply the idea of 'factoring' a system of PDE into a sequence of Cauchy problems that will be very important for applications.

A first-order system of PDE for s (dependent) functions u^a of n (independent) variables x^i

$$F\left(x^i,u^a,\frac{\partial u^a}{\partial x^i}\right) = 0 \tag{3.1}$$

can be thought of as defining a subset $X \subset \mathbb{R}^{n+s+ns}$. A reasonable condition to require would be that X should be a submanifold of $\mathbb{R}^{n+s+ns}$ of codimension $r \geq 0$ such that the projection $\pi : X \to \mathbb{R}^{n+s}$ is a submersion onto an open set. This captures the idea that, at least locally, one should be able to solve the equations (3.1) smoothly for r of the partial derivatives $\frac{\partial u^a}{\partial x^i}$ in terms of x, u and the other partial derivatives.

One can identify $\mathbb{R}^{n+s+ns}$ with coordinates (x^i,u_a,p^a_i) as the space $J^1(\mathbb{R}^n,\mathbb{R}^s)$ of '1-jets' of (germs of) maps from $\mathbb{R}^n$ to $\mathbb{R}^s$. Every differentiable mapping $f : \mathbb{R}^n \to \mathbb{R}^s$ then has a '1-graph' in $J^1(\mathbb{R}^n,\mathbb{R}^s)$ consisting of the points of the form

$$(x^i,u^a,p^a_i) = \left(x^1,\ldots,x^n,f^a(x^1,\ldots,x^n),\frac{\partial f^a}{\partial x^i}(x^1,\ldots,x^n)\right).$$

An important observation is that such a '1-graph' has the characterisic property that the so-called 'contact forms' on $J^1(\mathbb{R}^n,\mathbb{R}^s)$

$$\theta^a = \mathrm{d}u^a - p^a_i\,\mathrm{d}x^i \tag{3.2}$$

vanish when pulled back to a '1-graph'.

In his famous book *Partial Differential Relations*, M. Gromov calls a section of the bundle $\pi : J^1(\mathbb{R}^n,\mathbb{R}^s) \to \mathbb{R}^n$ *holonomic* if the forms θ^a vanish when pulled back via the section. It is a tautology that a holonomic section $\sigma : \mathbb{R}^n \to X \subset J^1(\mathbb{R}^n,\mathbb{R}^s)$ represents a solution of the PDE defined by X and conversely.

This suggests a generalization that turns out to be fundamental: Encoding PDE in terms of differential forms, which allows one to formulate the 'compatibility' conditions for a 'nested sequence of Cauchy problems' in geometric terms that only involve the exterior derivative.

3.1. **Differential ideals.** Let M^n be a smooth n-manifold. An *exterior differential system* on M is a graded, differentially closed ideal $\mathcal{I} \subset \mathcal{A}^*(M)$, the ring of differential forms on M.

While it is not strictly necessary, it simplifies some statements if one assumes that $\mathcal{I}$ is generated in positive degrees, i.e., $\mathcal{I}^0 = \mathcal{I} \cap \mathcal{A}^0(M) = (0)$, so I will assume this throughout the notes.

3.2. **Integral manifolds and elements.** An *integral manifold* of $\mathcal{I}$ is a submanifold $f : N \to M$ such that $f^*(\phi) = 0$ for all $\phi \in \mathcal{I}$.

Remark 2. In most applications of exterior differential systems, the integral manifolds of a certain dimension (often the maximal dimension) of a given differential ideal $\mathcal{I}$ represent the local solutions of some geometric problem that can be expressed in terms of partial differential equations. Thus, one is interested in techniques for describing the integral manifolds of a given $\mathcal{I}$.

An *integral element* of $\mathcal{I}$ is a p-plane $E \in \mathrm{Gr}_p(TM)$ such that $\iota_E^*(\phi) = 0$ for all $\phi \in \mathcal{I}$. The set of p-dimensional integral elements of $\mathcal{I}$ is a closed subset $\mathcal{V}_p(\mathcal{I}) \subseteq \mathrm{Gr}_p(TM)$. It is not always a smooth submanifold of this bundle.

Remark 3. Every tangent plane to an integral manifold of $\mathcal{I}$ is an integral element of $\mathcal{I}$. The fundamental problem in exterior differential systems is to decide, for a given $E \in \mathcal{V}_p(\mathcal{I})$, whether there is an integral manifold of $\mathcal{I}$ that has E as one of its tangent spaces.

Example 1 (The curl example revisited). The system (2.5) can be realized as an exterior differential system as follows: On $\mathbb{R}^6$ with coordinates $x^1, x^2, x^3, u_1, u_2, u_3$, consider the 2-form

$$\psi = \mathrm{d}(u_1\,\mathrm{d}x^1 + u_2\,\mathrm{d}x^2 + u_3\,\mathrm{d}x^3) - F^1(x,u)\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 - F^2(x,u)\,\mathrm{d}x^3 \wedge \mathrm{d}x^1 - F^3(x,u)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\,.$$

A mapping $u : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies the system (2.5) if and only if its graph in $\mathbb{R}^6$

$$(x,u) = \big((x^1,x^2,x^3,u_1(x^1,x^2,x^3),u_2(x^1,x^2,x^3),u_3(x^1,x^2,x^3)\big)$$

is an integral manifold of ψ. Note that the ideal generated by ψ is not usually differentially closed because

$$\mathrm{d}\psi = -\mathrm{d}F^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 - \mathrm{d}F^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^1 - \mathrm{d}F^3 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2$$

need not be a multiple of ψ. However, the ideal $\mathcal{I}$ generated by ψ and $\mathrm{d}\psi$ *is* differentially closed.

3.3. **Polar spaces.** While determining the structure of $\mathcal{V}_p(\mathcal{I})$ can be challenging, the problem of describing the $(p{+}1)$-dimensional integral elements that contain a given p-dimensional integral element is essentially a linear one.

Fix $E \in \mathcal{V}_p(\mathcal{I})$, with $E \subset T_xM$, and let $e_1, \ldots, e_p$ be a basis of E. The *polar space* (sometimes called the *enlargement space*) of E is the subspace

$$H(E) = \{ v \in T_xM \mid \phi(v, e_1, \ldots, e_p) = 0 \ \forall \phi \in \mathcal{I}^{p+1} \} \subset T_xM.$$

From its definition, any $E_+ \in \mathcal{V}_{p+1}(\mathcal{I})$ that contains E must be contained in $H(E)$ and, conversely, any $E_+ \in \mathrm{Gr}_{p+1}\big(H(E)\big)$ that contains E belongs to $\mathcal{V}_{p+1}(\mathcal{I})$. Set $c(E) = \dim\big(T_xM/H(E)\big)$.

3.4. **Cartan's Bound and characters.** Let $E \in \mathcal{V}_n(\mathcal{I})$ be fixed, and let $F = (E_0, E_1, \ldots, E_{n-1})$ be a flag of subspaces of E, with $\dim E_i = i$. Thus,

$$(0)_x = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E \subset T_xM.$$

Note that E_i belongs to $\mathcal{V}_i(\mathcal{I})$. The following result is due to Cartan and Kähler.

Proposition 1 (Cartan's Bound)**.** *Given $E \in \mathcal{V}_n(\mathcal{I})$ and a flag $F = (E_i)$ in E as above, there is an open E-neighborhood $U \subset \mathrm{Gr}_n(TM)$ such that $\mathcal{V}_n(\mathcal{I}) \cap U$ is contained in a smooth submanifold of U of codimension*

$$c(F) = c(E_0) + c(E_1) + \cdots + c(E_{n-1}).$$

Definition 1 (Regularity and Ordinarity). If $E \in \mathcal{V}_n(\mathcal{I})$ has a flag $F = (E_i)$ and an open neighborhood $U \subset \mathrm{Gr}_n(TM)$ such that $U \cap \mathcal{V}_n(\mathcal{I})$ is a smooth submanifold of U with codimension $c(F)$, one says that F is *regular* and that E is *ordinary.*[1]

Let $\mathcal{V}^o_n(\mathcal{I}) \subset \mathcal{V}_n(\mathcal{I})$ denote the subset consisting of ordinary integral elements of $\mathcal{I}$. It is an open (possibly empty) subset of $\mathcal{V}_n(\mathcal{I})$ that is a smooth submanifold of $\mathrm{Gr}_n(TM)$, and the basepoint projection $\pi : \mathcal{V}^o_n(\mathcal{I}) \to M$ is a submersion. (This uses my standing assumption that $\mathcal{I}^0 = (0)$.)

Remark 4 (Characters and Cartan's Test). A 'dual' version of Cartan's Bound (often called 'Cartan's Test') is useful: For any $E \in \mathcal{V}_n(\mathcal{I})$ and any flag $F = (E_0, E_1, \ldots, E_{n-1})$, the *character sequence* of F is the sequence of nonnegative integers $\bigl(s_0(F), s_1(F), \ldots, s_n(F)\bigr)$ such that

$$s_i(F) = \begin{cases} c(E_0) & i = 0, \\ c(E_i) - c(E_{i-1}) & 1 \le i < n, \\ \dim H(E_{n-1}) - n & i = n. \end{cases}$$

Then Cartan's Bound says that, near E, the subset $\mathcal{V}_n(\mathcal{I})$ is contained in a submanifold of $\mathrm{Gr}_n(TM)$ of dimension

$$\dim M + s_1(F) + 2s_2(F) + \cdots + ns_n(F).$$

Moreover, if, near E, the subset $\mathcal{V}_n(\mathcal{I})$ *is* a submanifold of $\mathrm{Gr}_n(TM)$ of this dimension, then E is ordinary and the flag F is regular.

When E is ordinary, the character sequence $\bigl(s_k(F)\bigr)$ is the same for all regular flags $F = (E_0, E_1, \ldots, E_{n-1})$ in E. This common sequence is known as the sequence of *Cartan characters* of E and simply written as the sequence $\bigl(s_k(E)\bigr)$. Moreover, the characters s_k are constant on the connected components of $\mathcal{V}^o_n(\mathcal{I})$.

4. Cartan-Kähler Theory

4.1. **A form of the Cartan-Kähler Theorem.** The main result needed in these notes is the following version of the Cartan-Kähler Theorem.[2]

Theorem 3 (Cartan-Kähler)**.** *Suppose that $\mathcal{I}$ is a real-analytic exterior differential system on M that is generated in positive degree and that $E \in \mathcal{V}_n(\mathcal{I})$ is ordinary. Then there exists a real-analytic integral manifold of $\mathcal{I}$ that has E as one of its tangent spaces.*

Remark 5 (Generality). The Cartan-Kähler theorem constructs the desired integral manifold by solving a sequence of initial value problems via the Cauchy-Kowalevski Theorem. At each step in the sequence, one gets to choose appropriate initial data that determine the resulting integral manifold. Examining the proof of the Cartan-Kähler theorem, one finds that there is an open E-neighborhood $U \subset \mathcal{V}_n(\mathcal{I})$ such that the initial data that determines a connected integral manifold of $\mathcal{I}$ whose tangent spaces belong to U consists of $s_0(E)$ constants, $s_1(E)$ functions of 1 variable, $s_2(E)$ functions of 2 variables, …, and $s_n(E)$ functions of n variables that are freely specifiable (i.e. 'arbitrary'), subject only to some open conditions.

[1] N.B.: When comparing other sources, the reader should be aware that what I call 'ordinary' in these notes is usually called 'Cartan-ordinary', which should not be confused with the weaker notion 'Kähler-ordinary', which will not be needed in these notes (so I will not discuss it).

[2] The standard, stronger version of the Cartan-Kähler Theorem has more technical hypotheses and so takes a bit longer to state. I won't need the stronger version in these notes.

Moreover, in the real-analytic setting, when the underlying manifold M is connected and the ideal $\mathcal{I}$ is generated in positive degree, the functions s_i are constant on the (necessarily smooth) submanifold $\mathcal{V}_n^o(\mathcal{I}) \subset \mathcal{V}_n(\mathcal{I})$ consisting of the ordinary integral elements of dimension n, and so one usually writes s_i instead of $s_i(E)$ (with n being understood, usually taken to be the maximal integer for which $\mathcal{V}_n^o(\mathcal{I})$ is not empty). In this case, (as we will see) the set $J_k^o(\mathcal{I})$ consisting of k-jets of ordinary integral manifolds (i.e., the ones whose tangent spaces are ordinary) is naturally a smooth manifold of dimension

$$N_k = n + s_0 + (k{+}1)\, s_1 + \binom{k+2}{2} s_2 + \cdots + \binom{k+n}{n} s_n\,.$$

Moreover, the natural projections $\pi_k^{k+1} : J_{k+1}^o(\mathcal{I}) \to J_k^o(\mathcal{I})$ are smooth submersions, surjective when $k \geq 1$. (Note that $J_1^o(\mathcal{I}) = \mathcal{V}_n^o(\mathcal{I})$.)

Thus, one often says that the ordinary integral manifolds of $\mathcal{I}$ 'depend on s_0 constants, s_1 functions of 1 variable, s_2 functions of 2 variables, ..., and s_n functions of n variables'.

References

[1] R. Bryant, *Notes on Exterior Differential Systems*, arXiv:1405.3116.

[2] É. Cartan, *Sur la structure des groupes inifinis de transformations*, Ann. Éc. Norm. **3** (1904), 153–206. (Reprinted in Cartan's Collected Works, Part II.)

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