LECTURE 3: EXAMPLES AND APPLICATIONS

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Abstract. These are my notes for Eilenberg Lecture 3. The goal of this lecture is to get some practice in computing with differential systems and then give some applications.

1. Computing polar spaces

If $\mathcal{I} \subset \mathcal{A}^*(M)$ is a graded exterior ideal, and $E \in \mathcal{V}_p(\mathcal{I})$ is a p-dimensional integral element with basis $e_1, \dots, e_p \in T_xM$, then the definition of the *polar space* $H(E)$ of E is

$$H(E) = \bigl\{ v \in T_xM \mid \kappa(e_1, \dots, e_p, v) = 0 \ \forall \kappa \in \mathcal{I}^{p+1} \ \bigr\},$$

and $c(E) = \dim\bigl(T_xM/H(E)\bigr)$ is the codimension of $H(E)$ in T_xM.

1.1. The algebraically finitely generated case.

In nearly all cases, $\mathcal{I}$ is generated *algebraically* by a finite number of differential forms $\kappa_1, \dots, \kappa_r$ where κ_j lies in $\mathcal{I}^{d_j+1}$, with $d_j \geq 0$. In this case, $H(E) \subset T_xM$ defined by the linear equations

$$\kappa_j(e_{i_1}, \dots, e_{i_{d_j}}, v) = 0 \quad \text{for } 1 \leq i_1 < \cdots < i_{d_j} \leq p, \quad \text{and } 1 \leq j \leq r.$$

So $c(E)$ is the dimension of the linear span of these '1-forms' in T^*_xM.

In practice, the number r of algebraic generators of $\mathcal{I}$ is usually small, and it is not difficult to compute the span of the above list of 1-forms, so this is often the quickest way to compute the number $c(E)$.

1.2. The Cartan characters.

For a given integral flag $F = (E_0, E_1, \dots, E_n)$ based at $x \in M$ we have

$$\{0_x\} = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n \subseteq H(E_n) \subseteq \cdots \subseteq H(E_i) \subseteq H(E_{i-1}) \subseteq \cdots \subseteq H(E_0) \subseteq T_xM$$

so

$$c(E_0) \leq c(E_1) \leq \cdots \leq c(E_n),$$

and it's useful to define the *Cartan characters* s_i of the flag F to be

$$s_i(F) = \begin{cases} c(E_0) & i = 0, \\ c(E_i) - c(E_{i-1}) & 1 \leq i < n, \\ \dim H(E_{n-1}) - n & i = n. \end{cases}$$

Note that $s_i \geq 0$ for all i.

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2. The Frobenius Theorem

Suppose that $\mathcal{I}$ on M^{n+s} can be locally generated *algebraically* by s linearly independent 1-forms $\theta^1, \dots, \theta^s$. In particular, since $\mathcal{I}$ is differentially closed, it follows that there are (local) 1-forms ϕ^a_b such that $\mathrm{d}\theta^a = \phi^a_b {\wedge} \theta^b$.

Now, there is a unique n-dimensional integral element at each point $x \in M$, namely the n-dimensional subspace $E_x \subset T_xM$ on which each of the θ^a vanish. Thus, $\mathcal{V}_n(\mathcal{I}) \subset \mathrm{Gr}_n(TM)$ is simply a copy of M, in fact, the image of a smooth section of the bundle $\mathrm{Gr}_n(TM)$, so it is a smooth manifold of dimension $n{+}s$. Meanwhile, for any flag $F = (E_0, \dots, E_n)$ in E_x, one has $H(E_p) = E_n$, so $c(E_p) = s$ for $0 \le p \le n$. In particular, $s_0(F) = s$ and $s_i(F) = 0$ for $0 < i < n$. Since $\dim \mathcal{V}_n(\mathcal{I}) = n{+}s = \dim M + s_1 + 2s_2 + \cdots + ns_n$, it follows that Cartan's Bound is saturated, and all of the elements of $\mathcal{V}_n(\mathcal{I})$ are ordinary and all their flags are regular.

By the Cartan-Kähler Theorem, every E_x is tangent to an integral manifold of $\mathcal{I}$ and the local integral manifolds near E depend on $s_0 = s$ constants.

Of course, in this particular case, there is another way to get the same result, which is to use the Frobenius Theorem (which is even better since it applies in the smooth setting). This Theorem says that, locally, it is possible to choose closed generators $\theta^a = \mathrm{d}y^a$ for some functions $y^1, \dots, y^s$ that form part of a coordinate system $x^1, \dots, x^n, y^1, \dots, y^s$. Then the local n-dimensional integral manifolds of $\mathcal{I}$ are the leaves defined by holding the y^a constant, so that the 'general' local n-dimensional integral manifold depends on s constants, in agreement with the prediction of the Cartan-Kähler Theorem.

3. A simple system

Fix integers $n > 0$ and $s > 0$ and use the index ranges $1 \le i, j, k \le n$ and $1 \le a, b \le s$. Consider a non-decreasing sequence of integers $0 \le c_0 \le c_1 \le \cdots \le c_n \le s$ and the system of PDE for functions u^a of the x^i of the form

$$\frac{\partial u^a}{\partial x^i} = 0 \quad \text{for } a \le c_{i-1}.$$

The general solution of this system is of the form

$$u^a = f^a(x^1, \dots, x^i) \qquad \text{for } c_{i-1} < a \le c_i$$

Explicitly, if $a \le c_0$ then u^a is constant, if $c_0 < a \le c_1$ then $u^a = f^a(x^1)$, etc. Thus, the Jacobian matrix $\frac{\partial u}{\partial x}$ is in 'generalized lower-triangular' form, with $c_0 + c_1 + \cdots + c_{n-1}$ zero entries 'above the diagonal'.

This can be reflected in the differential ideal $\mathcal{I}$ on $\mathbb{R}^{n+s}$ with coordinates $(x^1, \dots, x^n, u^1, \dots, u^s)$ by the differential forms

$$\Theta^a = \mathrm{d}u^a \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^i \qquad \text{for } c_{i-1} < a \le c_i\,.$$

Thus, when $c_{i-1} < a \le c_i$, the form Θ^a has degree $i{+}1$. Moreover, the ideal $\mathcal{I}$ is generated by the s forms Θ^a.

If we consider the integral element spanned at the origin $0 \in \mathbb{R}n+1$ by the vectors $e_i = \partial/\partial x^i$, and let E_i be the i-plane spanned by $e_1, \dots, e_i$, then it is easy to see that

$$H(E_i) = \{ v \in T_0\mathbb{R}^{n+s} \mid \mathrm{d}u^a(v) = 0 \text{ for } a \le c_i \},$$

so $c_i(E) = \dim\bigl(T_0\mathbb{R}^{n+s}/H(E_i)\bigr) = c_i$, as the notation suggests.

Since, by construction the n-plane spanned by

$$e_i(p) = \frac{\partial}{\partial x^i} + p^a_i \frac{\partial}{\partial u^a}$$

is an integral element of $\mathcal{I}$ for a given n-by-s matrix $p = (p^a_i)$ if and only if

$$p^a_i = 0 \quad \text{for } a \le c_{i-1},$$

it follows that $\mathcal{V}_n(\mathcal{I})$ has codimension equal to $c_0 + c_1 + \cdots + c_{n-1}$ in $\mathrm{Gr}_n(TM)$, so the flag $F = (E_0, E_1, \ldots, E_n)$ is regular.

Of course, in this particular case, we don't need the Cartan-Kähler Theorem to see the general solution or to check that the obvious sequence of Cauchy problems does indeed describe the general solution.

Note that, with the definition of the s_i given in Lecture 2, we have that the general solution depends on s_0 constants, s_1 functions of x^1, s_2 functions of (x^1, x^2), …, and s_n functions of $(x^1, \ldots, x^n)$.

Remark 1 (Linearity and Homogeneity). Note that if q^i and r^a are constants and $p = (p^a_i)$ are constants such that $p^a_i = 0$ when $a \leq c_j$ and $i > j$, then the affine mapping $\Phi : \mathbb{R}^{n+s} \to \mathbb{R}^{n+s}$ defined by

$$\Phi(x, u) = \bigl(xq,\ u + px + r\bigr)$$

satisfies $Phi^*(\Theta^a) = \Theta^a$, so it preserves the differential ideal $\mathcal{I}$ and hence preserves its integral elements and integral manifolds. This affine group acts transitively on the n-dimensional integral elements on which $\mathrm{d}x^1{\wedge}\cdots{\wedge}\mathrm{d}x^n$ does not vanish, so verifying that one such integral element is ordinary implies that they are all ordinary.

4. A example where Cartan-Kähler does not apply

4.1. **Two special cases.** Let $M = \mathbb{R}^3$, with coordinates x, y, z, and let $\mathcal{I}$ be generated by the 2-forms $\mathrm{d}x{\wedge}\mathrm{d}z$ and $\mathrm{d}y{\wedge}\mathrm{d}z$. Then the 2-plane field defined by $\mathrm{d}z = 0$ consists of 2-dimensional integral elements, and these are the only 2-dimensional integral elements, so that $\mathcal{V}_2(\mathcal{I})$ is a smooth 3-manifold in $\mathrm{Gr}_2(TM)$. Since $\mathcal{I}^1 = 0$, one has $c(E_0) = 0$ for all E_0. Letting E_1 be spanned by $a\partial_x + b\partial_y$, where $(a, b) \neq 0$, one finds that $H(E_1)$ has dimension 2 and is defined by $\mathrm{d}z = 0$, so $c(E_1) = 1$. Thus, $c(E_0) + c(E_1) = 1$ while the codimension of $\mathcal{V}_2(\mathcal{I})$ in $\mathrm{Gr}_2(TM)$ is 2. Thus, $E_2 = H(E_1)$ has no regular flag and hence is not ordinary.

It may seem disappointing that the Cartan-Kähler Theorem does not apply to prove the existence of 2-dimensional integral manifolds, especially, since there evidently does exist an integral manifold tangent to every 2-dimensional integral element of $\mathcal{I}$, namely, the plane $z = z_0$.

To see why one should not expect Cartan-Kähler to apply in this case, consider a modification of this example got by instead considering the ideal $\mathcal{I}'$ generated by $\mathrm{d}x{\wedge}\mathrm{d}z$ and $\mathrm{d}y{\wedge}(\mathrm{d}z - y\,\mathrm{d}x)$. The ideals $\mathcal{I}$ and $\mathcal{I}'$ are algebraically equivalent at each point, so there is a unique 2-dimensional integral element of $\mathcal{I}'$ through each point, namely the 2-plane that satisfies $\mathrm{d}z - y\,\mathrm{d}x = 0$. Since the dimensions of the polar spaces are the same for $\mathcal{I}'$ as they are for $\mathcal{I}$, these integral elements of $\mathcal{I}'$ are not ordinary, and this is just as well because there evidently are not any integral surfaces of the equation $\mathrm{d}z - y\,\mathrm{d}x = 0$.

Finally, note that this discussion applies more generally to the following situation: Consider a 3-manifold M endowed with a coframing $\omega = (\omega_1, \omega_2, \omega_3)$ and the ideal $\mathcal{I}$ generated by the pair of 2-forms $\{\omega_3{\wedge}\omega_1, \omega_3{\wedge}\omega_2\}$. Note that $\mathcal{I}$ is differentially closed for any such ω.

4.2. **A generalization.** Just as in the special cases considered before, the only 2-dimensional integral elements are the 2-planes satisfying $\omega_3 = 0$, and an integral flag $F = (E_0, E_1, E_2)$ satisfies $c(E_0) = 0$ and $c(E_1) = 1$, but the codimension of $\mathcal{V}_2(\mathcal{I})$ in $\mathrm{Gr}_2(TM)$ is $2 > c(E_0) + c(E_1)$. Hence there are no regular flags for $\mathcal{I}$.

Of course, we could consider the ideal $\mathcal{I}^+ \supset \mathcal{I}$ generated by ω_3 alone. This ideal has $c(E_0) = c(E_1) = 1$ for any integral flag $F = (E_0, E_1, E_2)$, so Cartan's Inequality is 'saturated':

$$\mathrm{codim}\bigl(\mathcal{V}_2(\mathcal{I}^+, \mathrm{Gr}_2(TM)\bigr) = 2 = c(E_0) + c(E_1)$$

Now, $\mathcal{I}^+$ will not be differentially closed unless $\mathrm{d}\omega_3$ is a multiple of ω_3, i.e., $\omega_3{\wedge}\mathrm{d}\omega_3 = 0$, so Cartan-Kähler would not apply except in this case. However, this is precisely the case where the Frobenius theorem applies, so Cartan-Kähler is not needed.

5. Lagrangians in Symplectic Geometry

Let $M = \mathbb{R}^{2n}$ and let $\mathcal{I}$ be generated by the symplectic form

$$\Omega = \mathrm{d}p_1 \wedge \mathrm{d}x^1 + \cdots + \mathrm{d}p_n \wedge \mathrm{d}x^n.$$

Then the n-plane E spanned by the ∂_{x^i} is an integral element of $\mathcal{I}$, and, if one takes the flag $F = (E_0, \ldots, E_{n-1})$ so that E_i is spanned by the ∂_{x^j} with $1 \le j \le i$, then one computes that, for $0 < i < n$, the polar space $H(E_i)$ is the subspace defined by $\mathrm{d}p_1 = \mathrm{d}p_2 = \cdots = \mathrm{d}p_i = 0$. Thus, $c(E_i) = i$.

By Cartan's Bound, $\mathcal{V}_n(\mathcal{I})$ has codimension at least

$$C = 1 + 2 + \cdots + (n-1) = \tfrac{1}{2}n(n-1)$$

in $\mathrm{Gr}_n(TM)$ near E. Meanwhile, any $\tilde{E} \in \mathrm{Gr}_n(TM)$ on which the $\mathrm{d}x^i$ are linearly independent will be defined by unique equations of the form

$$\mathrm{d}p_i - s_{ij}(\tilde{E})\,\mathrm{d}x^i = 0$$

for some numbers $s_{ij}(\tilde{E})$, and these functions s_{ij}, together with the x^i and the p_i define a local coordinate system on an open subset of $\mathrm{Gr}_n(TM)$ that contains E (which is defined by $s_{ij}(E) = 0$).

By Cartan's Lemma, such an $\tilde{E}$ will be an integral element of $\mathcal{I}$ if and only if $s_{ij}(\tilde{E}) - s_{ji}(\tilde{E}) = 0$. This is $\frac{1}{2}n(n-1)$ independent equations on $\tilde{E}$, so that $\mathcal{V}_n(\mathcal{I})$ has codimension $\frac{1}{2}n(n-1) = C$ in $\mathrm{Gr}_n(TM)$ near E. Consequently, E is ordinary, and the flag F is regular.

Of course, one already knows that Lagrangian manifolds exist, so this is not a surprise. Note, however, that what the Cartan-Kähler theorem would say is that one can specify an integral manifold on which the x^i are independent uniquely by choosing p_n to be an arbitrary function of the x^i, then choosing p_{n-1} subject to the condition that its partial in the x^n-direction equals the partial of p_n in the x^{n-1} direction (which determines p_{n-1} up to the addition of a function of $x^1, \ldots x^{n-1}$), then choosing p_{n-2} subject to the conditions that its partials in the x^n- and x^{n-1}-directions are determined by those of p_n and p_{n-1} (which determines p_{n-2} up to the addition of a function of $x^1, \ldots x^{n-2}$), etc. Thus, the integral manifolds are described by a choice of $1 = s_n(E)$ function of n variables, $1 = s_{n-1}(E)$ function of $n-1$ variables, etc., in agreement with the general theory.

Now, one can also specify a Lagrangian using only one function of n variables simply by taking $p_i = \frac{\partial u}{\partial x^i}$ for some function u of $x^1, \ldots, x^n$. However, for general $\mathcal{I}$, one cannot find such a formula that combines the 'arbitrary functions' in the general ordinary integral manifolds of $\mathcal{I}$ in this way.

Another way to interpret this 'discrepancy' is to note that the Lagrangian manifolds on which $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \neq 0$ are, by the above formula, put in correspondence with the arbitrary local function u of n variables, which, via its graph $\bigl(x^1, \ldots, x^n, u(x^1, \ldots, x^n)\bigr)$, is seen to be an integral manifold of the trivial ideal $\mathcal{I} = (0)$ on $\mathbb{R}^{n+1}$, which has Cartan characters

$$(s_0, s_1, \ldots, s_{n-1}, s_n) = (0, 0, \ldots, 0, 1).$$

Thus, this provides an example of the phenomenon that I mentioned earlier of two different exterior differential systems describing (local) solutions to the same problem. Note that they have the same last nonzero character, namely, $s_n = 1$, while their lower characters are not the same.

It is also instructive to note that, algebraically, all non-degenerate 2-forms on a vector space of dimension $2n$ are locally equivalent. Thus, if one considers *any* non-degenerate 2-form ω on $M = \mathbb{R}^{2n}$ and considers the algebraic ideal $\mathcal{I}$ generated by ω, then its maximal integral elements are n-planes $E_n \subset T_xM$ such that $E_n^*(\omega) = 0$. The codimension of $\mathcal{V}_n(\mathcal{I})$ in $\mathrm{Gr}_n(TM)$ is, again, $\frac{1}{2}n(n-1)$, and for *any* integral flag $F = (E_0, E_1, \ldots, E_n)$, one has $c_(E_i) = i$. Thus, all the integral flags are regular and every integral element $E_n \in \mathcal{V}_n(\mathcal{I})$ is ordinary. However, unless $\mathrm{d}\omega$ is in the algebraic ideal generated by ω, the ideal $\mathcal{I}$ is not differentially closed, so the Cartan-Kähler Theorem does not apply to $\mathcal{I}$.

6. Special Lagrangian Geometry

A slightly larger ideal than the Lagrangian ideal is the *special Lagrangian ideal*, which is generated on $M = \mathbb{C}^n = \mathbb{R}^{2n}$ by the 2-form

$$\omega = \frac{\mathrm{i}}{2}\left(\mathrm{d}z^1 \wedge \mathrm{d}\overline{z^1} + \cdots + \mathrm{d}z^n \wedge \mathrm{d}\overline{z^n}\right) = \mathrm{d}x^1 \wedge \mathrm{d}y^1 + \cdots + \mathrm{d}x^n \wedge \mathrm{d}y^n$$

and the n-form

$$\begin{aligned}\Upsilon &= \mathrm{Im}(\mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n) \\ &= \mathrm{d}y^1 \wedge \mathrm{d}x^2 \cdots \wedge \mathrm{d}x^n + \mathrm{d}x^1 \wedge \mathrm{d}y^2 \cdots \wedge \mathrm{d}x^n + \cdots + \mathrm{d}x^1 \wedge \mathrm{d}x^2 \cdots \wedge \mathrm{d}x^{n-1} \wedge \mathrm{d}y^n \\ &\qquad + \text{terms quadratic or higher in } \{\mathrm{d}y^1, \ldots, \mathrm{d}y^n\}\end{aligned}$$

The n-plane $E_n \subset T_0M$ defined by $\mathrm{d}y^1 = \cdots = \mathrm{d}y^n = 0$ is an integral element of the ideal $\mathcal{I}$ generated by ω and Υ. Computing with the flag $F = (E_0, E_1, \ldots, E_n)$ where E_i for $i > 0$ is spanned by $\partial/\partial x^1, \ldots \partial/\partial x^i$, one finds that for $i < n-1$ we have that

$$H(E_i) = \{v \in T_0M \mid \mathrm{d}y^j(v) = 0 \text{ for } 1 \le j \le i\},$$

while

$$H(E_{n-1}) = H(E_n) = E_n.$$

Thus, $c(E_i) = i$ for $i \le n-1$ while $c(E_{n-1}) = n$. Thus,

$$c(E_0) + \cdots + c(E_{n-1}) = 1 + 2 + \cdots + (n-2) + n = \tfrac{1}{2}n(n-1) + 1.$$

Meanwhile, an integral element near E_n is defined by equations

$$\mathrm{d}y^i - s^{ij}\,\mathrm{d}x^j = 0$$

where $s^{ij} - s^{ji} = 0$ (in order that ω vanish on the integral element while we must have

$$0 = \mathrm{Im}\big(\det(I_n + \mathrm{i}s)\big) = s^{11} + \cdots + s^{nn} + (\text{higher order terms in } s).$$

Thus, the space $\mathcal{V}_n(\mathcal{I})$ has codimension $\frac{1}{2}n(n-1) + 1$ in $\mathrm{Gr}_n(TM)$ near E_n, so Cartan's inequality is an equality, and the integral element E_n is ordinary.

The Cartan characters are $s_0 = 0$, $s_i = 1$ for $i < n-1$ and $s_{n-1} = 2$ while $s_n = 0$. The Cartan-Kähler Theorem then implies that the local special Lagrangian submanifolds depend on 2 functions of $n-1$ variables.

7. Isometric embedding of surfaces in Euclidean 3-space

This example will illustrate two things: First, that the 'natural' exterior differential system that one writes down to encode an interesting PDE may not satisfy the conditions of the Cartan-Kähler Theorem, but one may be able to 'augment' the system in a simple way to get a new system to which the Cartan-Kähler Theorem *does* apply. This will become a significant theme in future lectures.

Recall the the problem was, given a (real-analytic) Riemannian surface (M^2, g), prove that there exists (locally) an isometric immersion $u : M^2 \to \mathbb{R}^3$ satisfying $g = \mathrm{d}u \cdot \mathrm{d}u$.

Here is how we can set this up: The problem is local, so we can write $g = {\omega_1}^2 + {\omega_2}^2$ for two linearly independent 1-forms ω_1 and ω_2. The connection 1-form ω_{12} associated to this coframe satisfies

$$\begin{aligned}\mathrm{d}\omega_1 &= -\omega_{12} \wedge \omega_2\,, \\ \mathrm{d}\omega_2 &= \omega_{12} \wedge \omega_1\,, \\ \mathrm{d}\omega_{12} &= K\,\omega_1 \wedge \omega_2\,,\end{aligned}$$

where K is the Gauss curvature. We seek a mapping $u : M \to \mathbb{R}^3$ such that, when we write

$$\mathrm{d}u = v_1\,\omega_1 + v_2\,\omega_3$$

for some (unknown) $\mathbb{R}^3$-valued functions v_1 and v_2, we have

$$v_1 \cdot v_1 = v_2 \cdot v_2 = 1 \quad \text{while} \quad v_1 \cdot v_2 = 0.$$

Thus, v_1 and v_2 are orthonormal, and we can set $v_3 = v_1 \times v_2$, so that we will have an orthonormal triple. Thus, we can regard the ensemble (u, v_1, v_2, v_3) as a function with values in $\mathbb{R}^3 \times \mathrm{SO}(3)$.

So set $X^8 = M^2 \times \mathbb{R}^3 \times \mathrm{SO}(3)$ and define 1-forms $\eta_{ij} = -\eta_{ji}$ so that

$$\mathrm{d}(v_1\ v_2\ v_3) = (v_1\ v_2\ v_3) \begin{pmatrix} 0 & \eta_{12} & \eta_{13} \\ -\eta_{12} & 0 & \eta_{23} \\ -\eta_{13} & -\eta_{23} & 0 \end{pmatrix}.$$

Note that, by $d^2 = 0$, we must have

$$\mathrm{d}\eta_{12} = \eta_{13} \wedge \eta_{23}\,, \quad \mathrm{d}\eta_{13} = -\eta_{12} \wedge \eta_{23}\,, \quad \mathrm{d}\eta_{23}. = \eta_{12} \wedge \eta_{13}\,.$$

We will let $\mathcal{I}$ be the differential ideal generated by the three (linearly independent) components of $\theta = \mathrm{d}u - v_1\,\omega_1 - v_2\,\omega_3$. Now, from the structure equations, we see that

$$\begin{aligned} \mathrm{d}\theta &= -\mathrm{d}v_1 \wedge \omega_1 - v_1\,(-\omega_{12} \wedge \omega_2) - \mathrm{d}v_2 \wedge \omega_2 - v_2\,(\omega_{12} \wedge \omega_1) \\ &= (v_2\,\eta_{12} + v_3\,\eta_{13}) \wedge \omega_1 - v_2\,(\omega_{12} \wedge \omega_1) + (-v_1\,\eta_{12} + v_3\,\eta_{23}) \wedge \omega_2 + v_1\,(\omega_{12} \wedge \omega_2) \\ &= v_1\,(\omega_{12} - \eta_{12}) \wedge \omega_2 - v_2\,(\omega_{12} - \eta_{12}) \wedge \omega_1 + v_3\,(\eta_{13} \wedge \omega_1 + \eta_{23} \wedge \omega_2)\,. \end{aligned}$$

It follows that $\mathcal{I}$ is generated as an algebraic ideal by the three components of θ and the three 2-forms

$$\Upsilon_1 = (\omega_{12} - \eta_{12}) \wedge \omega_1\,, \quad \Upsilon_2 = (\omega_{12} - \eta_{12}) \wedge \omega_2\,, \quad \Upsilon_3 = \eta_{13} \wedge \omega_1 + \eta_{23} \wedge \omega_2\,.$$

We are looking for 2-dimensional integral manifolds of $\mathcal{I}$ on which $\omega_1{\wedge}\omega_2$ does not vanish, for such an integral manifold will be locally the graph of a mapping $(u, v_1, v_2, v_3) : M \to \mathbb{R}^3 \times \mathrm{SO}(3)$ such that $u : M \to \mathbb{R}^3$ is our desired isometric immersion.

In this case, $n = 2$ and $s = 6$. Consider an integral flag $F = (E_0, E_1, E_2)$ for which $\omega_1{\wedge}\omega_2$ does not vanish on E_2. Because the 1-forms in $\mathcal{I}$ are linear combinations of the three components of θ, it follows that $c(E_0) = 3$.

Because Υ_1 and Υ_2 must vanish on E_2, it follows that the 1-form $\omega 12 - \eta_{12}$ must vanish on E_2, and, hence, on E_1. Moreover, since Υ_3 must vanish on E_2, Cartan's Lemma says that there must be numbers $s_{ij} = s_{ji}$ such that E_2 is defined by the six equations

$$0 = \theta = \omega_{12} - \eta_{12} = \eta_{13} - s_{11}\,\omega_1 - s_{12}\,\omega_2 = \eta_{23} - s_{12}\,\omega_1 - s_{22}\,\omega_2\,.$$

Conversely, any choice of the three s_{ij} will define an integral element of $\mathcal{I}$ on which $\omega_1{\wedge}\omega_2$ is nonvanishing. Thus, we have

$$\operatorname{codim}\bigl(\mathcal{V}_2(\mathcal{I}), \mathrm{Gr}_2(TM)\bigr) = 12 - 3 = 9.$$

Now, E_1 will be spanned by a nonzero vector A such that $\theta(A) = 0$ and, say, $\omega_1(A) = a_1$, $\omega_2(A) = a_2$, where $(a_1, a_2) \neq (0, 0)$ (and, of course, $\theta(A) = (\omega_{12} - \eta_{12})(A) = 0)$. Thus,

$$\begin{aligned} A \lrcorner \Upsilon_1 &= a_1(\eta_{12} - \omega_{12}) \\ A \lrcorner \Upsilon_2 &= a_2(\eta_{12} - \omega_{12}) \\ A \lrcorner \Upsilon_3 &= -a_1\,\eta_{13} - a_2\,\eta_{23} + \eta_{13}(A)\,\omega_1 + \eta_{23}(A)\,\omega_2 \end{aligned}$$

Thus, we have $c(E_0) = 3$ and $c(E_1) = 5$. But

$$\operatorname{codim}\bigl(\mathcal{V}_2(\mathcal{I}), \mathrm{Gr}_2(TM)\bigr) = 12 - 3 = 9 > 3 + 5 = c(E_0) + c_1(E_1),$$

so Cartan's Inequality is strict, and the Cartan-Kähler Theorem does not apply.

However, since we must have $\omega_{12} - \eta_{12} = 0$ on any 2-dimensional integral manifold on which $\omega_1{\wedge}\omega_2 \neq 0$, we might as well consider the augmented ideal $\mathcal{I}^+$ generated by $\mathcal{I}$ plus the 1-form $\theta_0 = \eta_{12} - \omega_{12}$ and its exterior derivative

$$\mathrm{d}(\eta_{12} - \omega_{12}) = \eta_{13}\wedge\eta_{23} - K\,\omega_1\wedge\omega_2.$$

Thus, $\mathcal{I}^+$ is generated by four 1-forms, namely θ_0 and the three components of θ, and two 2-forms:

$$\Upsilon_3 = \eta_{13}\wedge\omega_1 + \eta_{23}\wedge\omega_2 \qquad \text{and} \qquad \mathrm{d}\theta_0 = \eta_{13}\wedge\eta_{23} - K\,\omega_1\wedge\omega_2.$$

So, now consider an integral flag $F = (E_0, E_1, E_2)$, where E_1 is spanned by a vector A such that $\theta(A) = \theta_0(A) = 0$ and

$$\omega_1(A) = a_1\,, \quad \omega_2(A) = a_2\,, \quad \omega_{13}(A) = b_1\,, \quad \omega_{23}(A) = b_2\,.$$

Then we have

$$A \lrcorner \Upsilon_3 = -a_1\,\eta_{13} - a_2\,\eta_{23} + b_1\,\omega_1 + b_2\,\omega_2\,,$$
$$A \lrcorner (\mathrm{d}\theta_0) = -b_2\,\eta_{13} + b_1\,\eta_{23} + Ka_2\omega_1 - Ka_1\omega_2\,.$$

Thus, the polar equations for E_1 will have rank 6 as long as $a_1b_1 + a_2b_2 \neq 0$, which will, generically, be true. In such a case, we will have $c(E_0) = 4$ and $c(E_1) = 6$.

Meanwhile, each 2-dimensional integral element of $\mathcal{I}^+$ satisfies

$$\theta = \theta_0 = \eta_{13} - s_{11}\omega_1 - s_{12}\omega_2 = \eta_{23} - s_{12}\omega_1 - s_{22}\omega_2 = 0$$

for some $s_{ij} = s_{ji}$ that satisfies the additional condition $s_{11}s_{22} - {s_{12}}^2 = K$. In particular, there is a 2-dimensional space of integral elements at every point of M, so that

$$\operatorname{codim}\bigl(\mathcal{V}_2(\mathcal{I}^+), \mathrm{Gr}_2(TM)\bigr) = 12 - 2 = 10 = 4 + 6 = c(E_0) + c_1(E_1),$$

so Cartan's Bound is saturated, and such integral elements are ordinary.

Thus, in the real-analytic category, every Riemannian surface can be locally isometrically embedded into $\mathbb{E}^3$.

Remark 2 (Higher isometric embedding). This argument actually generalizes to prove the *Cartan-Janet* isometric embedding theorem: Every real-analytic Riemannian n-manifold can be locally and real-analytically isometrically embedded in $\mathbb{R}^N$ for $N = \frac{1}{2}n(n{+}1)$.

Moreover, for special metrics in dimension n, one can prove isometric embedding in Euclidean spaces of dimension lower than $\frac{1}{2}n(n{+}1)$. Which leads to some interesting differential geometry problems that I will discuss later.

8. Cartan's Generalization of Lie's Third Theorem

I now turn to a more substantial application of the Cartan-Kähler Theorem. In fact, it was one of the motivations for Cartan to prove his original existence theorem in the first place. This will be a generalization of Lie's Third Fundamental Theorem, which asserts that every finite-dimensional Lie algebra is the Lie algebra of a (local) Lie transformation group.

Recall that, if G is a Lie group of dimension n and ω^i for $1 \leq i \leq n$ are a basis for the left-invariant 1-forms on G, then one has equations of the form

$$\mathrm{d}\omega^i = -\tfrac{1}{2}c^i_{jk}\,\omega^j\wedge\omega^k \tag{8.1}$$

for some unique constants $c^i_{jk} = -c^i_{kj}$. Since one has

$$0 = \mathrm{d}(\mathrm{d}\omega^i) = \tfrac{1}{6}\bigl(c^i_{mj}c^m_{kl} + c^i_{mk}c^m_{lj} + c^i_{ml}c^m_{jk}\bigr)\,\omega^j\wedge\omega^k\wedge\omega^l,$$

the constants c^i_{jk} satisfy the well-known quadratic equations

$$c^i_{mj}c^m_{kl} + c^i_{mk}c^m_{lj} + c^i_{ml}c^m_{jk} = 0$$

sometimes known as the *Jacobi identities*. Lie proved a converse to this result (now known as *Lie's Third Fundamental Theorem*), namely that, if a set of constants $c^i_{jk} = -c^i_{kj}$ with $1 \leq i, j, k \leq n$

satisfy the above Jacobi identities, then there exists a basis of left-invariant 1-forms ω^i on $\mathbb{R}^n$ that satisfy (8.1).

8.1. **Cartan's Theorem.** For many applications in differential geometry, Cartan needed a generalization of Lie's existence result, which he formulated as follows: Suppose that $C^i_{jk} = -C^i_{kj}$ and F^α_i (with $1 \le i, j, k \le n$ and $1 \le \alpha \le s$) are given functions on $\mathbb{R}^s$, and one wants to know whether or not there exist linearly independent 1-forms ω^i on $\mathbb{R}^n$ and a function $a = (a^\alpha) : \mathbb{R}^n \to \mathbb{R}^s$ that satisfy the *Cartan structure equations*

$$\mathrm{d}\omega^i = -\tfrac{1}{2} C^i_{jk}(a)\,\omega^j \wedge \omega^k \qquad \text{and} \qquad \mathrm{d}a^\alpha = F^\alpha_i(a)\,\omega^i. \tag{8.2}$$

Such a pair (a, ω) will be said to be an *augmented coframing* satisfying the structure equations (8.2).

Applying the fundamental identity $\mathrm{d}^2 = 0$ yields necessary conditions in order for such a pair (a, ω) to exist: One must have $\mathrm{d}\big(C^i_{jk}(a)\,\omega^j \wedge \omega^k\big) = \mathrm{d}(\mathrm{d}\omega^i) = 0$ and $\mathrm{d}\big(F^\alpha_i(a)\,\omega^i\big) = \mathrm{d}(\mathrm{d}a^\alpha) = 0$. Expanding these identities using (8.2) and the assumed independence of the ω^i then yields that, if, for each $u_0 \in \mathbb{R}^s$, an augmented coframing (a, ω) on some n-manifold M exists satisfying (8.2) with $a(x) = u_0$ for some $x \in M$, then one must have

$$F^\alpha_j \frac{\partial C^i_{kl}}{\partial u^\alpha} + F^\alpha_k \frac{\partial C^i_{lj}}{\partial u^\alpha} + F^\alpha_l \frac{\partial C^i_{jk}}{\partial u^\alpha} = \big(C^i_{mj} C^m_{kl} + C^i_{mk} C^m_{lj} + C^i_{ml} C^m_{jk}\big) \tag{8.3}$$

and

$$F^\beta_i \frac{\partial F^\alpha_j}{\partial u^\beta} - F^\beta_j \frac{\partial F^\alpha_i}{\partial u^\beta} = C^l_{ij}\, F^\alpha_l\,. \tag{8.4}$$

Cartan [2] proved a converse:

Theorem 1 (Cartan's Generalized Third Fundamental Theorem)**.** *Let $C^i_{jk} = -C^i_{kj}$ and F^α_i be real-analytic functions on $\mathbb{R}^s$ that satisfy* (8.3) *and* (8.4). *Then, for any $u_0 \in \mathbb{R}^s$, there exists an augmented coframing (a, ω) on $\mathbb{R}^n$ that satisfies* (8.2) *and has $a(0) = u_0$.* (*Moreover, any two such augmented coframings agree on a neighborhood of $0 \in \mathbb{R}^n$ up to a diffeomorphism of $\mathbb{R}^n$ that fixes $0 \in \mathbb{R}^n$.*)

Proof. Let $M = \mathbb{R}^n \times \mathrm{GL}(n, \mathbb{R}) \times \mathbb{R}^s$, and let $x : M \to \mathbb{R}^n$, $p : M \to \mathrm{GL}(n, \mathbb{R})$, and $u : M \to \mathbb{R}^s$ be the projections. Consider the ideal $\mathcal{I}$ generated on M by the n 2-forms

$$\Upsilon^i = \mathrm{d}(p^i_j\,\mathrm{d}x^j) + \tfrac{1}{2} C^i_{jk}(u)(p^j_l\,\mathrm{d}x^l) \wedge (p^k_m\,\mathrm{d}x^m)$$

and the s 1-forms

$$\theta^\alpha = \mathrm{d}u^\alpha - F^\alpha_i(u)\,(p^i_j\,\mathrm{d}x^j).$$

Note that one can write

$$\Upsilon^i = \pi^i_j \wedge \mathrm{d}x^j$$

for some 1-forms $\pi^i_j = \mathrm{d}p^i_j + P^i_{jk}\,\mathrm{d}x^k$ for some functions P^i_{jk} on M and that the forms π^i_j, $\mathrm{d}x^k$, and θ^α define a coframing on M, i.e., they are linearly independent everywhere and span the cotangent space everywhere.

Now, the hypothesis that $\mathrm{d}^2 = 0$ be a formal consequence of the structure equations (i.e., the equations (8.3) and (8.4)) is easily seen to be equivalent to the equations

$$\mathrm{d}\Upsilon^i = \tfrac{1}{2} \frac{\partial C^i_{jk}}{\partial u^\alpha}\,\theta^\alpha \wedge (p^j_l\,\mathrm{d}x^l) \wedge (p^k_m\,\mathrm{d}x^m) + C^i_{jk}\,\Upsilon^j \wedge (p^k_m\,\mathrm{d}x^m)$$

and

$$\mathrm{d}\theta^\alpha = \frac{\partial F^\alpha_i}{\partial u^\beta}\,\theta^\beta \wedge (p^i_j\,\mathrm{d}x^j) + F^\alpha_i\,\Upsilon^i.$$

Thus, these hypotheses imply that $\mathcal{I}$ is generated *algebraically* by the Υ^i and the θ^α. This makes it easy to choose an integral element and compute the Cartan characters:

Fix a point $z \in M$ and let $E \subset T_zM$ be the n-dimensional integral element defined by $\pi^i_j = \theta^\alpha = 0$. Let F be the flag in E defined so that E_i is also annihilated by the $\mathrm{d}x^j$ for $j > i$. Then one finds that $H(E_i)$ is defined by $\theta^\alpha = \pi^j_k = 0$ where $k \le i$, and hence that $c(E_i) = s + ni$ for $0 \le i \le n{-}1$. In particular, it follows that $\mathcal{V}_n(\mathcal{I})$ must be contained in a submanifold $\mathrm{Gr}_n(TM)$ of codimension at least $C = ns + \frac{1}{2}n^2(n{-}1)$.

Meanwhile, any n-plane $\tilde{E}$ on which the $\mathrm{d}x^i$ are linearly independent is specified by knowing the $ns + n^3$ numbers $s^\alpha_i(\tilde{E})$ and $s^i_{jk}(\tilde{E})$ such that $\tilde{E}$ satisfies

$$\pi^i_j - s^i_{jk}(\tilde{E})\,\mathrm{d}x^k = \theta^\alpha - s^\alpha_k(\tilde{E})\,\mathrm{d}x^k = 0.$$

The condition that such an $\tilde{E}$ be an integral element of $\mathcal{I}$ is then that $s^\alpha_k(\tilde{E}) = s^i_{jk}(\tilde{E}) - s^i_{kj}(\tilde{E}) = 0$, which is $ns + \frac{1}{2}n^2(n{-}1) = C$ equations on $\tilde{E}$. Thus, E is ordinary, and F is a regular flag.

Now, since the functions C^i_{jk} and F^α_i are assumed to be real-analytic, the Cartan-Kähler Theorem applies and one concludes that there is an integral manifold of $\mathcal{I}$ tangent to E. This integral manifold is described by having the p^i_j and the u^α be certain functions of the $x^1, \ldots, x^n$, say, $p^i_j = f^i_j(x)$ and $u^\alpha = a^\alpha(x)$. These then give the desired $(a^\alpha, \omega^i) = \bigl(a^\alpha(x), f^i_j(x)\,\mathrm{d}x^j\bigr)$. □

Remark 3 (Cartan's original theorem). The result just proved[1] is only a very special case of the theorem that Cartan proves in the first of his 'infinite groups' papers [2]. Later, we will see the full version, together with its application to PDE systems whose solutions depend on more than constants.

Remark 4 (Uniqueness). The general ordinary integral of $\mathcal{I}$ depends on n arbitrary functions of n variables (since the last nonzero character is $s_n = n$), but this is to be expected because, given any integral as a (local) coframing on $\mathbb{R}^n$, one can get others by simply pulling back by an arbitrary diffeomorphism of $\mathbb{R}^n$.[2] To get uniqueness up to local diffeomorphism for data (a, ω) in which a takes on a specific value $a_0 \in \mathbb{R}^s$, we will soon show that two such solutions are locally equivalent by an application of Cartan's technique of the graph.

Note, by the way, that when $s = 0$ (i.e., there are no functions a^α), this result becomes Lie's Third Theorem giving the existence of a local Lie group for any given Lie algebra.

8.2. **Smoothness and Globalization.** While this treatment assumes real-analyticity, so that the Cartan-Kähler Theorem can be applied, it is now known that the theorem is true in the smooth category as well. The proof in the smooth case is not difficult, but requires a little more insight than this simple application of Cartan-Kähler.

The reader will probably also have noticed that nothing is really used about the domain of the functions C^i_{jk} and F^α_i other than that it is a smooth manifold of some dimension s. This observation spurred the development of a 'globalized' version of Cartan's Theorem, which becomes the subject of *Lie algebroids*, in which $\mathbb{R}^s$ is replaced by a smooth manifold A. For details on these developments, as well as the smooth theory, the reader should consult treatises devoted to Lie algebroids, but I will sketch the translation here to aid in comparison with a somewhat generalized construction associated to a variant of Cartan's Theorem that I will describe in the next subsection.

Recall that a *Lie algebroid* is a manifold A endowed with a vector bundle $Y \to A$ of rank n whose space of sections $\Gamma(Y)$ carries a Lie algebra structure

$$\{,\}\colon \Gamma(Y) \times \Gamma(Y) \to \Gamma(Y)$$

[1]The proof in the text is Cartan's; I have merely simplified his proof as possible in this special case.

[2]Alternatively, one should think of the ordinary integral manifolds of $\mathcal{I}$ as giving a (local) augmented coframing (a, ω) satisfying the structure equations (8.2) *plus* a local coordinate system $x = (x^i)$ on the domain of (a, ω).

together with a bundle map $\alpha : Y \to TA$ that induces a homomorphism of Lie algebras on the spaces of sections[3] and that satisfies the Leibnitz compatibility condition

$$\{U, fV\} = \alpha(U)f\ V + f\{U, V\} \tag{8.5}$$

for all $U, V \in \Gamma(Y)$ and $f \in C^\infty(A)$.

A *realization* of $\bigl(A, Y, \{,\}, \alpha\bigr)$ is a triple (M, a, ω), where M is an n-manifold, $a : M \to A$ is a (smooth) mapping, and $\omega : TM \to a^*Y$ is a vector bundle isomorphism, such that $\alpha \circ \omega = \mathrm{d}a : TM \to TA$ and such that ω induces an isomorphism of Lie algebras on the space of sections of TM and a^*Y.

To see the translation from Cartan's language to that of Lie algebroids, start with the data of functions $C^i_{jk} = -C^i_{kj}$ and F^α_i on $A = \mathbb{R}^s$. Set $Y = A \times \mathbb{R}^n$ with a basis for sections U_i and set

$$\{U_i, U_j\} = C^k_{ij}\, U_k$$

and define $\alpha : Y \to TA = T\mathbb{R}^s$ by

$$\alpha(U_i) = F^\alpha_i \frac{\partial}{\partial u^\alpha}.$$

Then (8.3) and (8.4) are precisely the equations necessary and sufficient in order that (8.5) hold, that $\{,\}$ define a Lie bracket on the space of sections of Y, and that $\alpha : Y \to TA$ induce a homomorphism of Lie algebras.

Moreover, an augmented coframing (a^α, ω^i) on a manifold M^n satisfies Cartan's structure equations if and only if, when one sets

$$\omega = U_i\, \omega^i\,,$$

and defines $a : M \to \mathbb{R}^s$ to be $a = (a^\alpha)$, the data (M, a, ω) is a realization in the above sense.

This approach to globalizing Cartan's theorem has been very fruitful, and the reader is encouraged to consult the literature on Lie algebroids for more on this development.

However, it should be borne in mind that Cartan's original formulation in terms of what I am calling 'augmented coframings' turns out already to be very well suited for applications to differential geometry, as I hope to show in the discussion of examples in lectures to come.

References

[1] R. Bryant, *Notes on Exterior Differential Systems*, arXiv:1405.3116.

[2] É. Cartan, *Sur la structure des groupes inifinis de transformations*, Ann. Éc. Norm. **3** (1904), 153–206. (Reprinted in Cartan's Collected Works, Part II.)
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[3] Here, $\Gamma(TA)$, the set of vector fields on A, is given its standard Lie algebra structure via the Lie bracket.