LECTURE 4: UNIQUENESS IN CARTAN'S THEOREM AND EXAMPLES

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Abstract. These are my notes for Eilenberg Lecture 4. The goal of this lecture is to prove some basic results, including the local uniqueness in Cartan's Theorem, the (local) symmetry theorem, and to work out an interesting example.

1. Some lemmas

1.1. **A generalization of the Frobenius Theorem.** As we have seen, the Frobenius Theorem can be stated as follows: Given an $(n{+}s)$-manifold M and s linearly independent 1-forms $\theta^1, \dots \theta^s$ on M that algebraically generate a differentially closed ideal $\mathcal{I}$, then M is foliated by n-dimensional integral manifolds of $\mathcal{I}$.

For many applications, it's necessary to have a slight generalization of this result that takes into account functions as well as 1-forms.

Proposition 1. *Let $\mathcal{I}$ be a differentially closed ideal on a manifold M of dimension $n{+}s$ and suppose that $\mathcal{I}$ is locally generated algebraically by a finite number of functions $\{z^1, \dots, z^p\}$ together with s 1-forms $\{\theta^1, \dots, \theta^s\}$ that are linearly independent. Then the closed set*

$$Z = \{\, x \in M \,|\, z^\alpha(x) = 0 \text{ for } 1 \le \alpha \le p \,\} \tag{1.1}$$

is a disjoint union of n-dimensional integral manifolds of $\mathcal{I}$.

Proof. By the usual uniqueness theorems in ordinary differential equations, it suffices to show that every point of Z lies in at least one n-dimensional integral manifold of $\mathcal{I}$ since it is clear via the standard Frobenius Theorem argument that there is at most one n-dimensional integral manifold of $\mathcal{I}$ passing through each point of M.

Use the index ranges $1 \le a, b \le s$, $1 \le \alpha, \beta \le p$. By the differential closure of $\mathcal{I}$, there exist functions f^α_a, 1-forms ψ^α_β and ϕ^a_b, and 2-forms Υ^a_β on M satisfying

$$\begin{aligned} \mathrm{d}z^\alpha &= z^\beta\, \psi^\alpha_\beta + f^\alpha_b\, \theta^b \\ \mathrm{d}\theta^a &= z^\beta\, \Upsilon^a_\beta + \phi^a_b \wedge \theta^b \end{aligned} \tag{1.2}$$

I claim that any integral curve of the θ^a that intersects Z must lie entirely in Z. For suppose that $\gamma : [0,1] \to M$ satisfies $\gamma^*(\theta^a) = 0$ and $\gamma(0) = z \in Z$. Then the functions ζ^α on $[0,1]$ defined by $\zeta^\alpha(t) = z^\alpha\bigl(\gamma(t)\bigr)$ satisfy the initial conditions $\zeta^\alpha(0) = 0$ and the linear system of differential equations

$$\frac{\mathrm{d}\zeta^\alpha}{\mathrm{d}t} = \psi^\alpha_\beta\bigl(\gamma'(t)\bigr)\zeta^\beta, \tag{1.3}$$

which, by ODE uniqueness, forces $\zeta^\alpha(t) = 0$ for all t.

Fix $x \in Z$. By the linear independence of the θ^a, it follows that there is a neighborhood of z on which there exist vector fields X_i $(1 \le i \le n)$ that are linearly independent and satisfy $\theta^a(X_i) = 0$

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for all a and i. It then follows that there exists a smooth map L from a cubic neighborhood of $0 \in \mathbb{R}^n$ to M that satisfies

$$L(t^1, \dots, t^n) = \exp_{t^n X_n} \circ \cdots \circ \exp_{t^1 X_1}(z). \tag{1.4}$$

By shrinking the neighborhood of 0 if necessary, I arrange that L be a smooth embedding of the neighborhood into M. By construction, every curve of the form

$$\gamma(t) = L(x^1, \dots, x^{i-1}, t, 0, \dots, 0) \tag{1.5}$$

is an integral curve of X_i and hence of the 1-forms θ^a. By the argument above, the image of L lies entirely in the locus Z.

It remains to show that L is an integral manifold of the θ^a. To see this, set $\eta^a = L^*(\theta^a)$ and $\varphi^a_b = L^*(\phi^a_b)$. Then the structure equations above show that

$$\mathrm{d}\eta^a = \varphi^a_b \wedge \eta^b. \tag{1.6}$$

Also, the η^a vanish when pulled back via curves of the form $t \mapsto (x^1, \dots, x^{i-1}, t, 0, \dots, 0)$. Thus, when one writes $\eta^a = A^a_i \, dx^i$, the functions A^a_i satisfy

$$A^a_i(x^1, \dots, x^i, 0, \dots, 0) = 0 \tag{1.7}$$

and, by the equations for $\mathrm{d}\eta^a$, there are equations of the form

$$\frac{\partial A^a_i}{\partial x^j} - \frac{\partial A^a_j}{\partial x^i} = B^a_{bi} A^b_j - B^a_{bj} A^b_i \,. \tag{1.8}$$

Now the usual proof of the Frobenius Theorem applies: Uniqueness in a succession of ODE initial value problems shows that the A^a_i must vanish identically. Hence, $0 = \eta^a = L^*(\theta)$, so that the image of L is an integral manifold of the θ^a, as desired. □

Remark 1. Note that there is no claim that the closed set $Z \subset M$ is a manifold. Indeed, in most of the applications, Z is not a smooth manifold.

1.2. Uniqueness in Cartan's Theorem.

Our first application of this generalization of the Frobenius theorem is the following result, which proves the promised local uniqueness in Cartan's Theorem.

Theorem 1. *Suppose that, on a domain $D \subset \mathbb{R}^s$, there are specified smooth functions $C^i_{jk} = -C^i_{kj}$ and F^α_i, where the indices satisfy the ranges $1 \le i, j, k \le n$ and $1 \le \alpha \le s$. Suppose further that there are n-manifolds M and N, endowed with coframings ω and η, respectively, for which there exist smooth mappings $a : M \to D$ and $b : N \to D$ satisfying*

$$\begin{aligned} \mathrm{d}\omega^i &= -\tfrac12\, a^*C^i_{jk}\, \omega^j \wedge \omega^k & \qquad d\eta^i &= -\tfrac12\, b^*C^i_{jk}\, \eta^j \wedge \eta^k \\ \mathrm{d}a^\alpha &= a^*F^\alpha_j\, \omega^j & db^\alpha &= b^*F^\alpha_j\, \eta^j \end{aligned} \tag{1.9}$$

Then, for any $x \in M$ and $y \in N$ with $a(x) = b(y)$, there exists an x-neighborhood $U \subset M$ and a smooth map $f : U \to N$ satisfying $f(x) = y$, $f^*(\eta) = \omega$, and $f^*(b) = a$.

Proof. Let $a(x) = b(y) = a_0 \in D$. By a theorem of Whitney, in a neighborhood of (a_0, a_0) in $D \times D$ there exist smooth functions $F^\alpha_{i\beta}$ and $C^i_{jk\beta}$ so that

$$\begin{aligned} F^\alpha_i(p) - F^\alpha_i(q) &= F^\alpha_{i\beta}(p,q)(p^\beta - q^\beta) \\ C^i_{jk}(p) - C^i_{jk}(q) &= C^i_{jk\beta}(p,q)(p^\beta - q^\beta) \end{aligned} \tag{1.10}$$

for all (p, q) in this neighborhood. Thus, defining functions $z^\alpha = a^\alpha - b^\alpha$ on $M \times N$, there exist functions $H^\alpha_{i\beta}$ and $G^i_{jk\beta}$ on a neighborhood W of $(x, y) \in M \times N$ so that

$$\begin{aligned} a^*F^\alpha_i - b^*F^\alpha_i &= H^\alpha_{i\beta} z^\beta \\ a^*C^i_{jk} - b^*C^i_{jk} &= G^i_{jk\beta} z^\beta \end{aligned} \tag{1.11}$$

Setting $\theta^i = \eta^i - \omega^i$ on M and using the given structure equations, one obtains

$$\mathrm{d}z^\alpha = d(a^\alpha - b^\alpha) = a^*F^\alpha_i\,\omega^i - b^*F^\alpha_i\,\eta^i = H^\alpha_{i\beta}z^\beta\,\omega^i - b^*F^\alpha_i\,\theta^i. \tag{1.12}$$

Also,

$$\begin{aligned}\mathrm{d}\theta^i &= \mathrm{d}\eta^i - \mathrm{d}\omega^i = -\tfrac{1}{2}\,b^*C^i_{jk}\,\eta^i\wedge\eta^j + \tfrac{1}{2}\,a^*C^i_{jk}\,\omega^i\wedge\omega^j\\ &= -\tfrac{1}{2}\,b^*C^i_{jk}\left(\eta^i\wedge\eta^j - \omega^i\wedge\omega^j\right) + \tfrac{1}{2}G^i_{jk\beta}z^\beta\,\omega^i\wedge\omega^j\\ &= -\tfrac{1}{2}\,b^*C^i_{jk}\left(\theta^i\wedge\theta^j + \theta^i\wedge\omega^j + \omega^i\wedge\theta^j\right) + \tfrac{1}{2}G^i_{jk\beta}\,z^\beta\,\omega^i\wedge\omega^j.\end{aligned} \tag{1.13}$$

Thus, the ideal $\mathcal{I}$ generated by the θ^i and the z^α satisfies the hypotheses of Proposition 1 in a neighborhood of $(x,y)\in M\times N$. This then gives the existence of an n-dimensional integral manifold P of $\mathcal{I}$ in a neighborhood of (x,y).

By construction, the 1-forms ω^i are linearly independent on P, so, in a neighborhood of (x,y), P is the graph of a map $f: U\to N$ for some open neighborhood U of x that has the desired properties

$$f(x) = y,\quad f^*(\eta) = \omega,\quad\text{and}\quad f^*(b) = a. \tag{1.14}$$

(Note that $f: U\to f(U)$ is a diffeomorphism onto its image.) □

Remark 2. (The ‘finiteness’ hypothesis) The above theorem is not stated exactly this way in Cartan’s articles. Cartan’s original version went (roughly) as follows: If one has a coframing $\omega = (\omega^i)$ on an n-manifold M that satisfies structure equations of the form

$$\mathrm{d}\omega^i = -\tfrac{1}{2}C^i_{jk}\,\omega^j\wedge\omega^k, \tag{1.15}$$

the functions $C^i_{jk} = -C^i_{kj}$ are defined to be the (first-order) *differential invariants* of the coframing ω, and one defines the higher-order *derived differential invariants* sequentially as

$$\begin{aligned}\mathrm{d}C^i_{jk} &= C^i_{jkl}\,\omega^l,\\ \mathrm{d}C^i_{jkl} &= C^i_{jklm}\,\omega^m,\\ &\ \vdots\end{aligned} \tag{1.16}$$

Cartan observes that, if two coframings ω and η are equivalent under some diffeomorphism $f: M\to N$, i.e., $f^*\eta = \omega$, then the diffeomorphism f will also make the corresponding differential invariants match up. (This is because diffeomorphisms commute with the exterior derivative.)

He then says that, at some point, all the derived invariants at some order $K{+}1$ will be functions of some finite set $a^1,\dots,a^p$ of the derived invariants of order at most K, and then if two coframings ω and η both have this property and all of the derived invariants of order at most $K{+}1$ for both coframings are the ‘same functions of the same derived invariants’, then they will be equivalent under diffeomorphism.

The above statement is a more precise version of Cartan’s description, but it also makes explicit a ‘constant rank’ assumption that Cartan never discusses. Here is an example of the kind of problem that could show up, but that the hypotheses of the Theorem above avoid:

Let $M = \mathbb{R}^2$ and let $\omega^1 = dx$ and $\omega^2 = e^{f(x)}\,dy$ where f is a smooth function on $\mathbb{R}$ that vanishes identically on when $x \le 0$ and is positive when $x > 0$. Then the equations

$$\begin{aligned} \mathrm{d}\omega^1 &= 0\\ \mathrm{d}\omega^2 &= f'(x)\,\omega^1\wedge\omega^2\\ \mathrm{d}\bigl(f'(x)\bigr) &= f''(x)\,\omega^1\\ &\ \ \vdots\\ \mathrm{d}\bigl(f^{(k)}(x)\bigr) &= f^{(k+1)}(x)\,\omega^1\\ &\ \ \vdots \end{aligned} \tag{1.17}$$

show that the derived invariant functions are simply the derivatives of f. Note that, although two points (x_0, y_0) and $(0, y_1)$ with $x_0 < 0$ have the same derived invariants at the point in question, the coframing ω does not admit a local diffeomorphism preserving the coframing and taking (x_0, y_0) to $(0, y_1)$.

Corollary 1. *Suppose that, on a domain $D \subset \mathbb{R}^s$, there are specified real-analytic functions $C^i_{jk} = -C^i_{kj}$ and F^α_i, where the indices satisfy the ranges $1 \le i, j, k \le n$ and $1 \le \alpha \le s$. Then any augmented coframing $(a, \omega) = (a^\alpha, \omega^i)$ on an n-manifold M that satisfies*

$$\mathrm{d}\omega^i = -\tfrac{1}{2}\,a^*C^i_{jk}\,\omega^j\wedge\omega^k \quad \text{and} \quad \mathrm{d}a^\alpha = a^*F^\alpha_j\,\omega^j \tag{1.18}$$

is real-analytic in some local coordinates.

1.3. **Symmetries of coframings.** Let M be a connected n-manifold endowed with a coframing $\omega : TM \to \mathbb{R}^n$. Kobayashi showed that the set $\mathrm{Aut}(M, \omega)$ consisting of the diffeomorphisms $f : M \to M$ that satisfy $f^*(\omega) = \omega$ is a Lie group. In fact, for any $m \in M$, the evaluation map $\mathrm{ev}_m : \mathrm{Aut}(M, \omega) \to M$ defined by $\mathrm{ev}_m(f) = f(m)$ embeds $\mathrm{Aut}(M, \omega)$ into M as a closed submanifold.

Kobayashi's theorem generalizes to the case of symmetries of augmented coframings (a, ω) on a manifold M. Moreover, there is a 'constant rank' theorem for the mapping $a : M \to \mathbb{R}^s$ that is an important first step in proving the smooth version of Cartan's Theorem: Namely, the map $a : M \to \mathbb{R}^s$ has constant rank, implying that the fibers are smooth submanifolds of M of locally constant dimension.

Proposition 2. *Let $C^i_{jk} = -C^i_{kj}$ and F^α_i be smooth functions on $\mathbb{R}^s$ for $1 \le i, j, k \le n$ and $1 \le \alpha, \beta \le s$ that satisfy the equations*

$$F^\alpha_j\frac{\partial C^i_{kl}}{\partial u^\alpha} + F^\alpha_k\frac{\partial C^i_{lj}}{\partial u^\alpha} + F^\alpha_l\frac{\partial C^i_{jk}}{\partial u^\alpha} = \bigl(C^i_{mj}C^m_{kl} + C^i_{mk}C^m_{lj} + C^i_{ml}C^m_{jk}\bigr) \tag{1.19}$$

and

$$F^\beta_i\frac{\partial F^\alpha_j}{\partial u^\beta} - F^\beta_j\frac{\partial F^\alpha_i}{\partial u^\beta} = C^l_{ij}\,F^\alpha_l\,. \tag{1.20}$$

Let M be a 1-connected n-manifold, and let $(a, \omega) = (a^\alpha, \omega^i)$ be an augmented coframing on M that satisfies the structure equations

$$\mathrm{d}\omega^i = -\tfrac{1}{2}C^i_{jk}(a)\,\omega^j\wedge\omega^k \qquad \text{and} \qquad \mathrm{d}a^\alpha = F^\alpha_i(a)\,\omega^i. \tag{1.21}$$

Then the rank of the s-by-n matrix $F = F^\alpha_i(a)$ is a constant r on M. Moreover, the Lie algebra consisting of the symmetry vector fields of (a, ω) has dimension equal to $n-r$.

Proof. Recall that a vector field Y on M is a symmetry vector field of (a, ω) if it satisfies

$$\mathsf{L}_Y\, a = 0 \quad \text{and} \quad \mathsf{L}_Y\, \omega = 0. \tag{1.22}$$

Such a vector field Y is determined by the functions $y^i = \omega^i(y)$, which must satisfy the conditions (which are linear in y)

$$\mathsf{L}_Y(\omega^i) = \mathrm{d}(\omega^i(Y)) + Y \lrcorner\, \mathrm{d}\omega^i = \mathrm{d}y^i - C^i_{jk}(a) y^j\, \omega^k = 0 \tag{1.23}$$

and

$$\mathsf{L}_Y(a^\alpha) = F^\alpha_i(a) y^i = 0. \tag{1.24}$$

This suggests the following construction. Let $P = M \times \mathbb{R}^n$ with projection $v = (v^i) : P \to \mathbb{R}^n$, and set

$$z^\alpha = F^\alpha_i(a) v^i \quad \text{and} \quad \theta^i = \mathrm{d}v^i - C^i_{jk}(a) v^j\, \omega^k. \tag{1.25}$$

Let $\mathcal{I}$ denote the exterior ideal on P generated algebraically by the s functions z^α and the n (linearly independent) 1-forms θ^i. The equations (1.19) and (1.20) imply that $\mathcal{I}$ is differentially closed, so Proposition 1 implies that the locus $Z \subset P$ on which the z^α vanish is a union of n-dimensional integral manifolds of $\mathcal{I}$.

We can assume that $F = \big(F^\alpha_i(a)\big)$ does not identically vanish, otherwise the mapping $a : M \to \mathbb{R}^s$ and, hence, the $C^i_{jk}(a)$, are constants, which is the local uniqueness of Lie's Theorem.

Consider a point $m \in M$ where the rank of $F\big(a(m)\big) = \big(F^\alpha_i(a(m))\big)$ is $r_m > 0$. Clearly, there is an open m-neighborhood $U \subset M$ on which the rank of $F(a)$ is at least r_m. Then there exist $n - r_m$ linearly independent vectors $w_l \in \mathbb{R}^n$ for $1 \le l \le n - r_m$ such that $F(a(m))w_l = 0$. Let Y_l denote the symmetry vector field that satisfies $\omega(Y_l(m)) = w_l$. These Y_l exist by the above argument and must be linearly independent on some open m-neighborhood $U' \subset U$, and they must also satisfy $F(a)\omega\big(Y_l\big) \equiv 0$ on U', implying that $F(a)$ cannot have rank greater than r_m on U'. Thus the rank of $F(a)$ must be locally constant. Since M is connected, the rank of $F(a)$ must be constant.

Consequently, the locus $Z \subset P = M \times \mathbb{R}^n$ is a subbundle of the trivial bundle $M \times \mathbb{R}^n$ and the equations

$$\nabla v = \left(\mathrm{d}v^i - \big(C^i_{jk}(a)\omega^k\big)\, v^j\right) = 0 \tag{1.26}$$

define a connection on Z that on a neighborhood of every point of M has a basis of parallel sections. Consequently, this connection is flat. Since M is simply-connected, it follows that there is a global basis of flat sections of the connection, i.e., a global basis of symmetry vector fields on M for the augmented coframing (a, ω). □

2. An extended example

I want to return to one of the first examples I discussed. The goal is to study those Riemannian surfaces (M^2, g) whose Gauss curvature K satisfies the second order system

$$\mathrm{Hess}_g(K) = a(K)g + b(K)\mathrm{d}K^2$$

for some functions a and b of one variable.

Writing $g = {\omega_1}^2 + {\omega_2}^2$ on the orthonormal frame bundle F^3 of M, the structure equations become

$$\begin{aligned} \mathrm{d}\omega_1 &= -\omega_{12} \wedge \omega_2 & \mathrm{d}\omega_{12} &= K\, \omega_1 \wedge \omega_2 \\ \mathrm{d}\omega_2 &= \omega_{12} \wedge \omega_1 & \mathrm{d}K &= K_1\, \omega_1 + K_2\, \omega_2 \end{aligned}$$

and the condition to be studied is encoded as

$$\begin{pmatrix} \mathrm{d}K_1 \\ \mathrm{d}K_2 \end{pmatrix} = \begin{pmatrix} -K_2 \\ K_1 \end{pmatrix} \omega_{12} + \begin{pmatrix} a(K) + b(K)\, {K_1}^2 & b(K)\, K_1 K_2 \\ b(K)\, K_1 K_2 & a(K) + b(K)\, {K_2}^2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Applying $\mathrm{d}^2=0$ to these two equations yields

$$\big(a'(K)-a(K)b(K)+K\big)\,K_i=0 \qquad \text{for } i=1,2.$$

Thus, unless $a'(K)=a(K)b(K)-K$, such metrics have K constant.

Conversely, suppose that $a'(K)=a(K)b(K)-K$. The question becomes 'Does there exist a 'solution' (F^3,ω) to the following system?'

$$\begin{aligned}\mathrm{d}\omega_1 &= -\omega_{12}\wedge\omega_2\\ \mathrm{d}\omega_2 &= \omega_{12}\wedge\omega_1 \qquad\qquad \omega_1\wedge\omega_2\wedge\omega_{12}\neq 0,\\ \mathrm{d}\omega_{12} &= K\,\omega_1\wedge\omega_2\end{aligned}$$

where

$$\begin{pmatrix}\mathrm{d}K\\ \mathrm{d}K_1\\ \mathrm{d}K_2\end{pmatrix}=\begin{pmatrix}K_1 & K_2 & 0\\ a(K)+b(K)\,{K_1}^2 & b(K)\,K_1K_2 & -K_2\\ b(K)\,K_1K_2 & a(K)+b(K)\,{K_2}^2 & K_1\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_{12}\end{pmatrix}.$$

Since $\mathrm{d}^2=0$ is formally satisfied for these structure equations, Cartan's Theorem applies and guarantees that, for any constants (k,k_1,k_2), there is a local solution with the invariants (K,K_1,K_2) taking the value (k,k_1,k_2).

In fact, the above equations show that, on a solution, the $\mathbb{R}^3$-valued function (K,K_1,K_2) either has rank 0 (if $K_1=K_2=a(K)=0$) or rank 2. Moreover, one sees that

$$-\big(a(K)+b(K)({K_1}^2{+}{K_2}^2)\big)\,\mathrm{d}K+K_1\,\mathrm{d}K_1+K_2\,\mathrm{d}K_2=0,$$

so that the image of a connected solution lies in an integral leaf of this 1-form, which only vanishes when $K_1=K_2=a(K)=0$. Setting $L={K_1}^2{+}{K_2}^2$, this expression becomes

$$-2\big(a(K)+b(K)L\big)\,\mathrm{d}K+\mathrm{d}L=0,$$

which has an integrating factor: If $\lambda(K)$ is a nonzero solution to $\lambda'(K)=-b(K)\lambda(K)$, then

$$-2\lambda(K)^2a(K)\,\mathrm{d}K+\mathrm{d}\big(\lambda(K)^2L\big)=0,$$

so that the curvature map has image in a level set of the function

$$F(K,K_1,K_2)=\lambda(K)^2({K_1}^2{+}{K_2}^2)-\mu(K),$$

where $\mu'(K)=2\lambda(K)^2a(K)$. (This function has critical points only where $K_1=K_2=a(K)=0$.)

On any solution (F^3,ω), the vector field Y defined by the equations

$$\omega_1(Y)=\lambda(K)K_2,\quad \omega_2(Y)=-\lambda(K)K_1,\quad \omega_{12}(Y)=\lambda(K)a(K),$$

is a symmetry vector field of the coframing (since the Lie derivative of each of ω_1, ω_2, ω_{12} with respect to Y is zero). It is nonvanishing on a solution of rank 2, and, up to constant multiples, it is the unique symmetry vector field of the coframing on any connected solution.

For simplicity, I will only consider the case $b(K)\equiv 0$ in the remainder of this discussion. In this case, $a'(K)=-K$, so $a(K)=\frac{1}{2}(c-K^2)$ for some constant c and $\lambda'(K)=0$, so one can take $\lambda(K)\equiv 1$.

The most interesting case is when $c>0$, and, by scaling the metric g by a constant, one can reduce to the case $c=1$. Thus, the equations simplify to

$$\begin{pmatrix}\mathrm{d}K\\ \mathrm{d}K_1\\ \mathrm{d}K_2\end{pmatrix}=\begin{pmatrix}K_1 & K_2 & 0\\ \frac{1}{2}(1{-}K^2) & 0 & -K_2\\ 0 & \frac{1}{2}(1{-}K^2) & K_1\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_{12}\end{pmatrix}.$$

and these functions satisfy

$$F(K,K_1,K_2)={K_1}^2{+}{K_2}^2+\tfrac{1}{3}K^3-K=C$$

where C is a constant (different from the previous c, which is now normalized to 1).

There are two critical points of F, namely $(K, K_1, K_2) = (\pm 1, 0, 0)$, and these correspond to the surfaces whose Gauss curvature is identically $+1$ or identically -1. These clearly exist globally so it remains to consider the other level sets.

The level sets with $C < -\frac{2}{3}$ are connected and contractible, in fact, they can be written as graphs of K as a function of ${K_1}^2+{K_2}^2$. $C = -\frac{2}{3}$ contains the critical point $(K, K_1, K_2) = (1, 0, 0)$, but away from this point, it is also a smooth graph. When $-\frac{2}{3} < C < \frac{2}{3}$, the level set has two smooth components, a compact 2-sphere that encloses the critical point $(1, 0, 0)$ and a graph of K as a smooth function of ${K_1}^2+{K_2}^2$. The level set $C = \frac{2}{3}$ is singular at the point $(-1, 0, 0)$, but, minus this point, it has two smooth pieces, one bounded and simply connected, and one unbounded and diffeomorphic to $\mathbb{R} \times S^1$. For $C > \frac{2}{3}$, the level set is connected and contractible.

According to the general theory, for each contractible component L of a (smooth part of a) level set $F = C$, there will exist a simply-connected solution manifold (F^3, ω) whose curvature image is L and whose symmetry vector field Y is complete. Moreover, the time-2π-flow of the vector field X_{12} (i.e., the vector field that satisfies $\omega_1(X_{12}) = \omega_2(X_{12}) = 0$ while $\omega_{12}(X_{12}) = 1$) is a symmetry of the coframing ω and hence is the time-T-flow of Y for some $T > 0$. Dividing F by the $\mathbb{Z}$-action that this generates produces a solution manifold $(\bar{F}, \omega)$ that is no longer simply-connected but on which the flow of X_{12} is 2π-periodic, and this is the necessary and sufficient condition that $\bar{F}$ be the oriented orthonormal frame bundle of a Riemannian surface (M^2, g) satisfying the desired equation.

However, for the components of the level sets that are diffeomorphic to the 2-sphere, this global existence result does not generally hold, i.e., the corresponding solution manifolds (F^3, ω) need not be the orthonormal frame bundles of complete Riemannian surfaces (M^2, g). I will explain why for the 2-sphere components of the level sets $F = \epsilon^2 - 2/3$ where $\epsilon > 0$ is small.

Suppose that a connected solution manifold (F^3, ω) whose curvature map has, as image, such a 2-sphere component is found and that the symmetry vector field Y as defined above is complete on it. Then the metric $h = {\omega_1}^2 + {\omega_2}^2 + {\omega_{12}}^2$ must be complete on F. Now, for small positive ϵ, one has that K is close to 1 while K_1 and K_2 are close to zero, so it follows from a computation that the sectional curvatures of h are all positive. In particular, the completeness of the metric on F^3 implies, by Bonnet-Meyers, that it is compact, with finite fundamental group.

By passing to a finite cover, one can assume that F is simply connected. I claim that the symmetry vector field Y has closed orbits and that its flow generates an S^1-action on F. To see this, note that the map $(K, K_1, K_2) : F \to \mathbb{R}^3$ submerses onto the 2-sphere leaf. Hence the fibers over the two points where $K_1 = K_2 = 0$ must be a finite collection of circles that are necessarily integral curves of the vector field Y, which has no singular points. In particular, the flow of Y on one of these circles must be periodic, but, because the flow of Y preserves the coframing ω, if some time $T > 0$ flow of Y has a fixed point, then the time T flow of Y must be the identity. Thus, the flow of Y is periodic with some minimal positive period $T > 0$, so it generates a free S^1-action on F. The quotient by this free S^1-action is a connected quotient surface that is a covering of the 2-sphere. Since this covering must be trivial, the orbits of Y are the fibers of the map (K, K_1, K_2) to the 2-sphere. In particular, F, being connected and simply-connected, must be diffeomorphic to the 3-sphere.

Now, consider the vector field X_{12} on F as defined above. This vector field is (K, K_1, K_2)-related to the vector field

$$-K_2 \frac{\partial}{\partial K_1} + K_1 \frac{\partial}{\partial K_2}$$

on $\mathbb{R}^3$ whose flow is rotation about the K-axis with period 2π.

It also follows that the flow of X_{12} preserves the two circles that are defined by $K_1 = K_2 = 0$. If (F, ω) is to be a covering of the orthonormal frame bundle of a Riemannian surface (M^2, g), then X_{12} must be periodic of period $2k\pi$ for some integer $k > 0$. As already remarked, by the structure equations, the 2π-flow of X_{12}, say Ψ, is a symmetry of the coframing and hence must be the time $R > 0$ flow of Y for some unique $R \in (0, T]$.

Now, along each of the two circles in F defined by $K_1 = K_2 = 0$, one has $Y = a(K)X_{12} \neq 0$. The two points where $K_1 = K_2 = 0$ satisfy $K = K_\pm(\epsilon)$ where $K_-(\epsilon) < 1 < K_+(\epsilon)$ and $\frac{1}{3}K_\pm(\epsilon)^3 - K_\pm(\epsilon) = \epsilon^2 - \frac{2}{3}$. In fact, one finds expansions

$$K_\pm(\epsilon) = 1 \pm \epsilon - \tfrac{1}{6}\epsilon^2 \pm \tfrac{5}{72}\epsilon^3 - \cdots$$

and this implies that

$$a\big(K_\pm(\epsilon)\big) = \tfrac{1}{2}(1 - K_\pm(\epsilon)^2) = \mp\epsilon - \tfrac{1}{3}\epsilon^2 + \cdots .$$

Thus the ratios of X_{12} to Y on these two circles are not equal or opposite, and hence Y cannot have the same period on these two circles, which is impossible. Thus, there cannot be a global solution surface for such a leaf.

References

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