Finsler Manifolds with Constant Flag Curvature

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I. Finsler Geometry

A Finsler structure on M^n is a smooth hypersurface in TM

$$\begin{array}{ccc} \Sigma & \hookrightarrow & TM \\ & & \downarrow \pi \\ & & M \end{array}$$

with the properties:

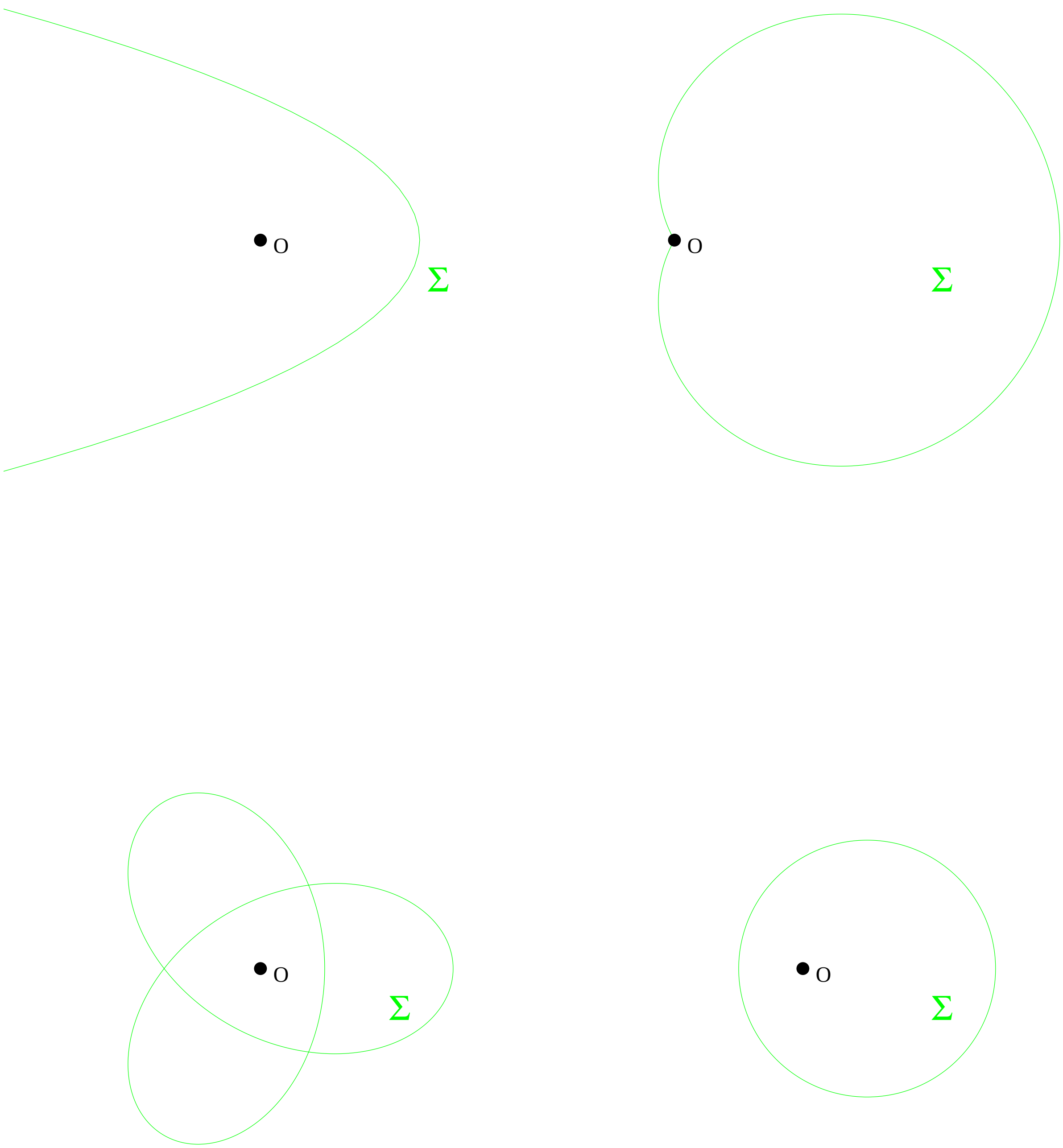
i. $\pi : \Sigma \longrightarrow M$ is a submersion,

ii. $\Sigma_x = \Sigma \cap T_x M$ is strictly convex towards $0_x \in T_x M$ for all $x \in M$.

A Σ-curve or unit speed curve is a curve $\gamma : [a, b] \longrightarrow M$ with $\gamma'(t) \in \Sigma$ for almost all $t \in [a, b]$. Define its length to be

$$L(\gamma) = b - a.$$

Possible Σ_x when $n = 2$



Example: Zermelo Navigation

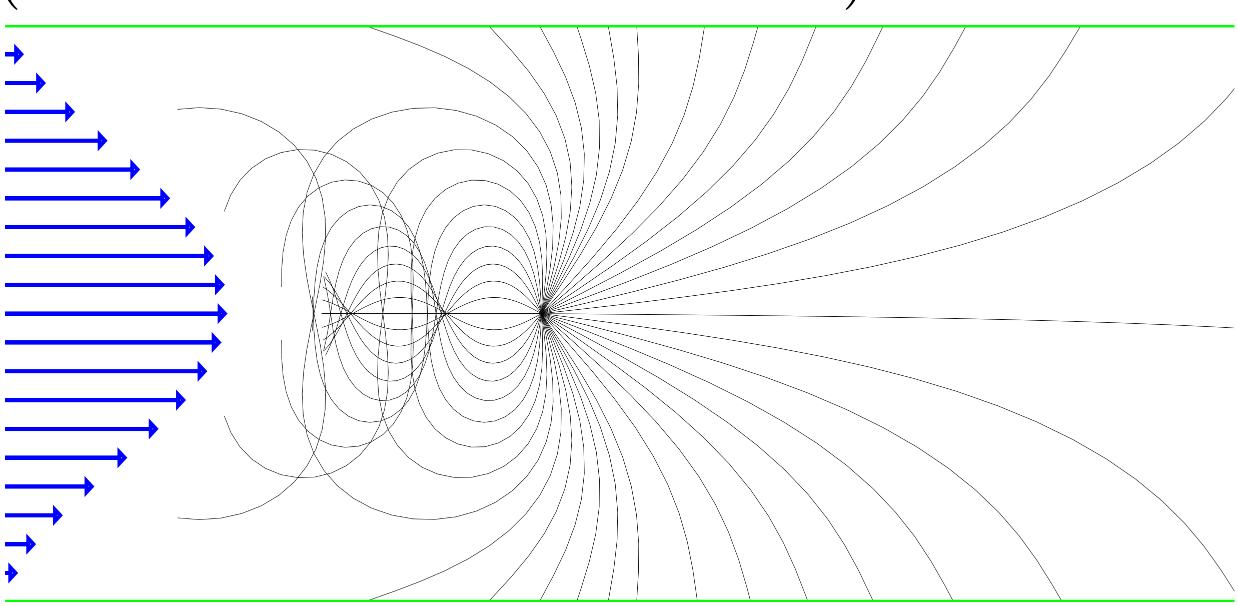
M is a domain in the plane and C is a vector field on M with $|C(p)| < 1$ for all $p \in M$. Set

$$\Sigma_p = \left\{ v + C(p) \,\middle|\, |v| = 1 \right\}.$$

A 'river': M is defined by $|y| < 1$. Set

$$C(x, y) = 0.8(1 - y^2)^2 \, \frac{\partial}{\partial x}$$

(current is faster in mid-river).

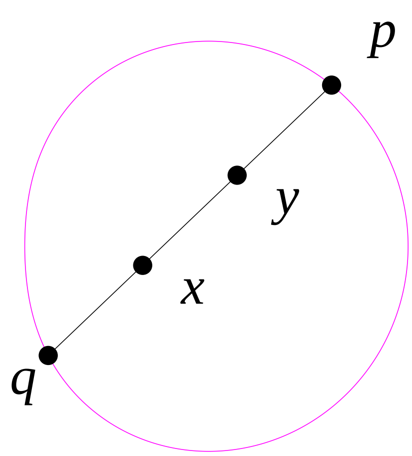


Example: Hilbert's Projective Metric

$M \subset \mathbb{R}^n$ is a convex domain.

For $x, y \in M$ define

$$d(x,y) = \frac{1}{2}\log\left(\frac{(p-x)\,(y-q)}{(p-y)\,(x-q)}\right).$$



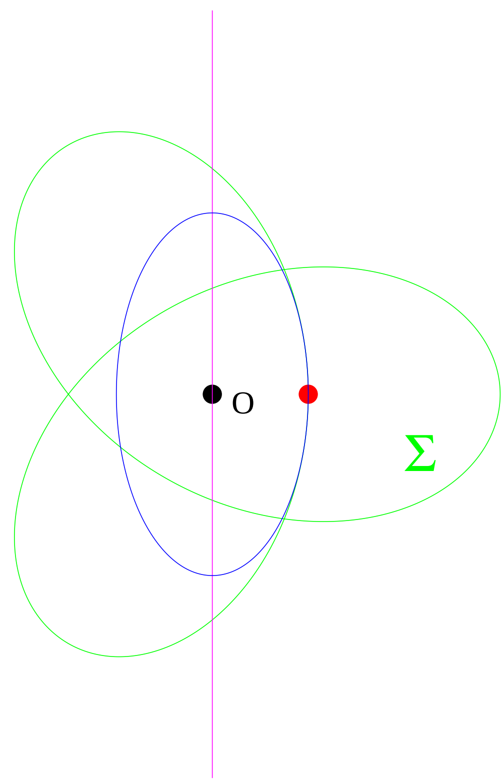
This is the distance function of a Finsler structure and its geodesics are line segments. This is the usual hyperbolic metric if M is the interior of an ellipsoid.

Constructions at $x \in M$

For $u \in \Sigma_x$ define $u^* \in T_x^*M$ so that $u^* = 1$ defines the tangent plane to Σ_x at u. Set $u^\perp = \ker u^*$ and let S_u be the ellipsoid centered on 0_x that best approximates Σ_x at u.

$$T_xM = \mathbb{R}u \oplus u^\perp$$

$$T_u\Sigma_x \simeq u^\perp$$



S_u is the unit sphere of a quadratic form g_u on T_xM, and

$$g_u = (u^*)^2 + h_u$$

Geodesics:

The *Hilbert* 1-*form* ω on Σ is defined by

$$\omega_u = \pi^*\big(u^*\big).$$

Since $\omega\wedge\big(d\omega\big)^{n-1} \neq 0$, there is a *Reeb vector field* E on Σ satisfying

$$\omega\big(E\big) = 1, \qquad E \lrcorner\, d\omega = 0.$$

E generates the geodesic flow on Σ.

Set $\eta_u = \pi^*(h_u)$ and $\eta' = \mathfrak{L}_E\eta$. Then η' has type $(n{-}1, n{-}1)$. There is a unique splitting

$$\ker\omega_u = T_u\Sigma_x \oplus H_u$$

so that H_u is null for η' and Lagrangian for $d\omega$.

Direct sum decomposition: Recall that $T_u\Sigma_x \simeq u^\perp \simeq H_u$, so there is a canonical metric on Σ, so that the sum is orthogonal, with $ds_h^2 = \eta$, etc.

$$\begin{array}{ccccccccc}
ds^2 & = & \omega^2 & + & ds_h^2 & + & ds_v^2 \\
T_u\Sigma & = & \mathbb{R}\,E_u & \oplus & H_u & \oplus & T_u\Sigma_x \\
\big\downarrow{\scriptstyle \pi'(u)} & & \big\downarrow & & \big\downarrow & \swarrow & \\
T_xM & = & \mathbb{R}\,u & \oplus & u^\perp & &
\end{array}$$

Defines an $O(n)$-structure on Σ.

Theorem: All local invariants of Σ can be reconstructed from $\left(\omega, ds^2\right)$.

Proof: See works by Cartan, Berwald, Chern, and Foulon.

Jacobi Operators and Flag Curvature

For $u \in \Sigma$, there is a self-adjoint operator

$$K(u) : u^{\perp} \longrightarrow u^{\perp}$$

so that if $\gamma : [a, b] \to M$ a Σ-geodesic and $\gamma' : [a, b] \to \Sigma$ is its tangential lift, then the Jacobi operator of γ is

$$Jf(s) = \nabla^2 f(s) + K\big(\gamma'(s)\big)\, f(s)$$

for $f \in \Gamma\big((\gamma')^{\perp}\big)$. Say $K(u)v$ is the curvature of the 'flag' (u, v). [In general, $K(u)v \neq K(v)u$, even when this makes sense.]

Σ has constant flag curvature c if

$$K(u)v = c\, v$$

for all $u \in \Sigma$ and $v \in u^{\perp}$.

Example: Hilbert's Projective Metric has constant flag curvature $c \equiv -1$. (Funk, 1929, for $n = 2$. Buseman for $n > 2$?)

Order of Invariants

In the Riemannian case, $\Sigma \subset TM$ is given by $\frac{1}{2}n(n{+}1)$ functions of n variables.

- ω is of order zero,
- E is first order,
- H, ds^2 are first order,
- K is second order.

The general Finsler structure $\Sigma \subset TM$ is given by one function of $2n{-}1$ variables.

- ω is first order,
- E is second order,
- H, ds^2 are third order,
- K is fourth order.

II. Finsler Surfaces

M^2 oriented, $\Sigma^3 \subset TM$ a Finsler structure.

Cartan's Structure Equations:

$$\begin{aligned} d\omega_1 &= -\omega_2{\wedge}\omega_3 \\ d\omega_2 &= -\omega_3{\wedge}\omega_1 + I\,\omega_3{\wedge}\omega_2 \\ d\omega_3 &= -K\,\omega_1{\wedge}\omega_2 + J\,\omega_3{\wedge}\omega_2 \end{aligned}$$

($\omega = \omega_1$ and $E = E_1$.)

$I = J = 0$ defines the Riemannian case, in which K is just the Gauss curvature and $\omega_3 = \omega_{21}$ is the Levi-Civita connection form.

The Bianchi Identities:

$$\begin{aligned} dI &= J\,\omega_1 + I_2\,\omega_2 + I_3\,\omega_3 \\ dJ &= -(K_3{+}KI)\,\omega_1 + J_2\,\omega_2 + J_3\,\omega_3 \\ dK &= K_1\,\omega_1 + K_2\,\omega_2 + K_3\,\omega_3 \end{aligned}$$

Global Rigidity for $K = -1$

Theorem (Akbar-Zadeh, 1988) *If Σ is compact and $K \equiv -1$, then Σ is Riemannian.*

Proof: Σ and hence M compact implies the geodesic flow is complete. Geodesics lift to Σ as the integral curves of E_1 and, along such a curve, with arclength parameter s, Bianchi says

$$\frac{dI}{ds} = J \qquad\qquad \frac{dJ}{ds} = I\,.$$

Thus,

$$I(s) = c_1\, e^{s} + c_2\, e^{-s},$$

but I is bounded, so $c_1 = c_2 = 0$. Thus $I = 0$ on all curves, so $I = 0$, which forces $J = 0$ (Bianchi again), which forces Σ to be Riemannian.

A Double Fibration

Assume $K \equiv +1$ and Σ compact. Then geodesics leaving $p \in M$ focus at length π, just as in the Riemannian case, and M is either $\mathbb{RP}^2$ or S^2. Assume $M = S^2$. Then geodesics close at length 2π. The space of geodesics is $\Lambda \simeq S^2$, with projection $\lambda : \Sigma \to \Lambda$, yielding the double fibration:

$$\begin{array}{ccccc} & & \Sigma & & \\ & \lambda \swarrow & & \searrow \pi & \\ \Lambda & & & & M \end{array}$$

There is a metric g with area form dA and 1-form φ on Λ so that

$$\lambda^* g = {\omega_2}^2 + {\omega_3}^2$$

$$\lambda^* dA = \omega_3 \wedge \omega_2 = d\omega_1$$

$$\lambda^* \varphi = I\,\omega_2 + J\,\omega_3$$

These quantities satisfy

$$d\varphi = (1 - R)\, dA$$

where R is the Gauss curvature of g.

Every $p \in M$ defines a curve $C_p = \lambda\big(\pi^{-1}(p)\big)$, the curve of geodesics passing through p. This curve satisfies

$$\kappa_g\, ds_g = \varphi|_C\,.$$

Such curves are the (g, φ)-geodesics on Λ.

Reconstruction Theorem: *If (g, φ) are a metric and 1-form on $\Lambda = S^2$ with the property that $d\varphi = (1 - R)\, dA$ and that the (g, φ)-geodesics are closed, then there is a unique Finsler structure Σ with $K = 1$ on the space M of such curves whose geodesic data is (g, φ).*

Zoll-type Examples:

If u is any smooth function on S^2 then the (g, φ)-geodesics are the same as the $(\tilde{g}, \tilde{\varphi})$-geodesics, where

$$\tilde{g} = e^{2u}\, g\,, \qquad\qquad \tilde{\varphi} = \varphi + * du\,.$$

If g is a metric on S^2 with positive curvature $R > 0$, then taking $u = \frac{1}{2} \log R$ yields a pair $(\tilde{g}, \tilde{\varphi} = * du)$ with

$$d\tilde{\varphi} = (1 - \tilde{R})\, \widetilde{dA}.$$

By a theorem of Guillemin, there are many Zoll metrics g on S^2 with positive curvature, so this gives many Finsler structures on S^2 with $K = 1$, since $(g, 0)$-geodesics are just the ordinary geodesics of g.

Projectively Flat Examples

Regard S^2 as $\mathbb{S}$, oriented lines thru $0 \in \mathbb{R}^3$. If $\mathbf{v} = (v_0, v_1)$ is a pair of vectors in $\mathbb{R}^3$ with $v_0 \wedge v_1 \neq 0$, define $\gamma_{\mathbf{v}} : S^1 \to \mathbb{S}$ by

$$\gamma_{\mathbf{v}}(s) = [\,\cos s\, v_0 + \sin s\, v_1\,]_+.$$

If $\mathbf{w} = (w_0, w_1)$, then $\gamma_{\mathbf{w}}$ parametrizes the same oriented great circle as $\gamma_{\mathbf{v}}$ iff $[v_0 \wedge v_1]_+ = [w_0 \wedge w_1]_+$. They have the same speed iff

$$[\![v_0 + i\, v_1]\!] = [\![w_0 + i\, w_1]\!] \in \mathbb{CP}^2 \setminus \mathbb{RP}^2$$

Theorem: (Funk) *If $\Sigma \subset T\mathbb{S}$ is a Finsler structure with $K = 1$ and geodesics the great circles, then every geodesic has a unit speed parametrization of the form $\gamma_{\mathbf{v}}$.*

If $\mathbb{S}^*$ is the space of oriented great circles in $\mathbb{S}$, a Finsler structure on $\mathbb{S}$ with great circle geodesics and $K = 1$ induces a section $\sigma : \mathbb{S}^* \to \mathbb{CP}^2 \setminus \mathbb{RP}^2$ of the bundle map

$$\begin{array}{ccc} \mathbb{CP}^2 \setminus \mathbb{RP}^2 & \longrightarrow & \mathbb{S}^* \\ \\ [\![v_0 + i\, v_1]\!] & \longmapsto & [v_0 {\wedge} v_1]_+ \end{array}$$

Theorem: *The image* $\sigma\big(\mathbb{S}^*\big) \subset \mathbb{CP}^2 \setminus \mathbb{RP}^2$ *is a (holomorphic) conic. Conversely, if* $C \subset \mathbb{CP}^2 \setminus \mathbb{RP}^2$ *is a smooth conic, then it is* $\sigma\big(\mathbb{S}^*\big)$ *for some unique Finsler structure* Σ *on* $\mathbb{S}$ *with great circle geodesics and* $K = 1$*. Such Finsler structures* Σ_1 *and* Σ_2 *on* $\mathbb{S}^2$ *are isometric if and only if the corresponding* C_1 *and* C_2 *are equivalent under* $\mathrm{SL}(3, \mathbb{R})$*.*

Moduli: Any conic without real points in $\mathbb{CP}^2$ is $\mathrm{SL}(3,\mathbb{R})$-equivalent to a unique conic of the form

$$z_0{}^2 + e^{ip_1}\, z_1{}^2 + e^{ip_2}\, z_2{}^2 = 0,$$

where $0 \leq p_1 \leq p_2 < \pi$. Thus, the moduli space $\mathcal{M}$ is of dimension 2.

- $\mathcal{M}$ is not compact.
- The unit spheres Σ_x for $x \in \mathbb{S}$ are usually quartic plane curves
- Except for the Riemannian case ($p = q = 0$), these are not Zoll-derived examples.

III. Higher Dimensions Analog in higher dimensions: Regard S^n as $\mathbb{S}$, oriented lines thru $0 \in \mathbb{R}^{n+1}$. If $\mathbf{v} = (v_0, v_1)$ is a pair of vectors in $\mathbb{R}^n$ with $v_0 \wedge v_1 \neq 0$, define $\gamma_{\mathbf{v}} : S^1 \to \mathbb{S}$ by

$$\gamma_{\mathbf{v}}(s) = [\, \cos s\, v_0 + \sin s\, v_1\,]_+.$$

If $\mathbf{w} = (w_0, w_1)$, then $\gamma_{\mathbf{w}}$ parametrizes the same oriented great circle as $\gamma_{\mathbf{v}}$ iff $[v_0 \wedge v_1]_+ = [w_0 \wedge w_1]_+$. They have the same speed iff

$$[\![v_0 + i\, v_1]\!] = [\![w_0 + i\, w_1]\!] \in \mathbb{CP}^n \setminus \mathbb{RP}^n$$

Theorem: (Buseman?) *If $\Sigma \subset T\mathbb{S}$ is a Finsler structure with $K \equiv 1$ and great circle geodesics, then every geodesic has a unit speed parametrization of the form $\gamma_{\mathbf{v}}$.*

If $\mathbb{Q}$ is the space of oriented great circles in $\mathbb{S}$, a Finsler structure on $\mathbb{S}$ with great circle geodesics and $K \equiv 1$ induces a section $\sigma : \mathbb{Q} \to \mathbb{CP}^n \setminus \mathbb{RP}^n$ of the bundle map

$$\begin{array}{ccc} \mathbb{CP}^n \setminus \mathbb{RP}^n & \longrightarrow & \mathbb{Q} \\ [\![v_0 + i\, v_1]\!] & \longmapsto & [v_0 {\wedge} v_1]_+ \end{array}$$

Theorem: *The image $\sigma\bigl(\mathbb{Q}\bigr) \subset \mathbb{CP}^n \setminus \mathbb{RP}^n$ is a (holomorphic) quadric. Conversely, if $Q \subset \mathbb{CP}^n \setminus \mathbb{RP}^n$ is a smooth quadric, then it is $\sigma\bigl(\mathbb{Q}\bigr)$ for some unique Finsler structure Σ on $\mathbb{S}$ with great circle geodesics and $K \equiv 1$. Such Finsler structures Σ_1 and Σ_2 on $\mathbb{S}^2$ are isometric if and only if the corresponding Q_1 and Q_2 are equivalent under $\mathrm{SL}(n{+}1, \mathbb{R})$.*

Moduli: Any quadric without real points in $\mathbb{CP}^n$ is $\mathrm{SL}(n+1, \mathbb{R})$-equivalent to a unique conic of the form

$$z_0{}^2 + e^{ip_1}\, z_1{}^2 + \cdots + e^{ip_n}\, z_n{}^2 = 0,$$

where p_i are real numbers satisfying

$$0 \leq p_1 \leq \cdots \leq p_n < \pi.$$

The moduli space $\mathcal{M}_n$ is of dimension n and is not compact.

General case of constant flag curvature c:

If the geodesic space Λ exists and is Hausdorf, it has dimension $2(n-1)$ and we have a double fibration

$$\begin{array}{ccccc} & & \Sigma & & \\ & \lambda \swarrow & & \searrow \pi & \\ \Lambda & & & & M \end{array}$$

When $c \neq 0$, there is a (pseudo-)metric g and a symplectic form Ω on Λ so that

$$\lambda^* g = c\, ds_h^2 + ds_v^2$$
$$\lambda^* \Omega = d\omega$$

Remarkable Fact: $\nabla^g \Omega = 0.$

Corollary: The holonomy of g lies in

$$\begin{array}{ll} \mathrm{GL}(n-1, \mathbb{R}) \subset O(n-1, n-1), & \text{if } c < 0, \\ U(n-1) \subset O\big(2(n-1)\big), & \text{if } c > 0. \end{array}$$