FINSLER SURFACES WITH PRESCRIBED CURVATURE CONDITIONS

Robert L. Bryant

Duke University

July 8, 1995

Dedicated to the Nankai Institute of Mathematics on the occasion of its tenth anniversary.

Abstract. A generalization of Finsler structures on surfaces is proposed and the differential invariants of such structures are developed. The information obtained is then used to construct examples of Finsler structures and generalized Finsler structures which satisfy various interesting curvature conditions. In particular, examples are constructed of non-Riemannian Finsler structures on the 2-sphere whose 'Gaussian curvature' is constant and of non-Riemannian complete Finsler structures on the plane whose 'Gaussian curvature' is a negative constant. The local and global generality of the Finsler structures satisfying geometrically natural conditions is discussed using É. Cartan's method of exterior differential systems.

0. Introduction

This is an expository manuscript on the geometry of Finsler surfaces. My goal in writing it was two-fold. First, I wanted to explain the construction of some new examples of Finsler surfaces, examples with certain curvature properties which had been sought for some time. Second, I wanted to explain some of the general methods from exterior differential systems that I had found useful in the construction, with the hope that others might find these methods useful as well. As part of this second goal, I also wanted to provide a fairly complete example of the use of the ideas of exterior differential systems that could be used as a resource for those interested in learning more about exterior differential systems. As such, I hope that this manuscript can be read with profit by those who are not necessarily motivated by applications to Finsler geometry *per se*. Of course, if it does stimulate the reader's interest in Finsler geometry, then so much the better. However, an exposition of

1991 *Mathematics Subject Classification.* 53B40, 53A55, 53C60, 53C22, 58G30, 58E25.

Key words and phrases. Finsler geometry, differential invariants, geodesic flows.

This work was begun during a visit to the Nankai Institute of Mathematics in May 1995. My thinking on Finsler geometry has benefited enormously from conversations with Shiing-Shen Chern and Zhongmin Shen that took place during that visit. The support of the Nankai Institute and from the National Science Foundation in the form of Grant DMS-9205222 is hereby gratefully acknowledged.

A shorter, less expository version of this manuscript will be submitted to the proceedings of the 1995 AMS Summer Workshop on Finsler Geometry.

Finsler geometry is not one of my main goals, so the reader will find that I have not treated any of the basic Finsler material in any depth. Those wishing to learn more about Finsler geometry should consult some of the works listed in the references that are specifically devoted to this subject.

Roughly speaking, a Finsler structure on an n-manifold M is a choice of a Banach norm on each of the tangent spaces to M, the choice being required to 'vary smoothly with the basepoint' in the appropriate sense. The familiar Riemannian case is simply the case where each of the Banach norms is the norm induced by a quadratic form. Geometrically, the same information is specified by choosing a 'unit sphere' in each tangent space, so that one arrives at a smooth hypersurface $\Sigma \subset TM$ which has the property that each fiber $\Sigma_m = \Sigma \cap T_m M$ is a smooth, strictly convex hypersurface in $T_m M$ which surrounds the origin $0_m \in T_m M$. Given one such hypersurface $\Sigma \subset TM$, any appropriately small perturbations of Σ in the C^2 sense will also define a Finsler structure on M, so, intuitively, the Finsler structures near a given one 'depend' on a choice of one function of $2n{-}1$ variables. This has to be contrasted with Riemannian structures on M, which, by the same sort of analysis, are seen to depend on $\binom{n+1}{2}$ arbitrary functions of n variables, the coefficients of the quadratic form in local coordinates.

Given a Finsler structure on M, one can define the notion of unit speed curve and the notion of distance between points, which is the infimum of the length of unit speed curves joining the two points. Thus, much of metric Riemannian geometry can be carried over to Finsler geometry. However, the problem of defining parallel transport and curvature turns out to be more subtle than in the Riemannian case, in large part because a Finsler structure is not defined as a section of some finite dimensional tensor bundle over M, instead it is somehow tensorial on the unit sphere bundle Σ. This motivated Cartan [Ca2] and, later, Chern [Ch1] to analyse Finsler geometry from the point of view of a natural G-structure with connection on the manifold Σ. They showed that there is a natural $\mathrm{O}(n{-}1)$-structure on Σ and that one could choose one of several possible natural connections on Σ, any one of which allowed one to generalize the differential geometric curvature constructions carried out in Riemannian geometry. For example, curvature tensors appear which play the role in the second variation of arc length that is played by the Ricci tensor in Riemannian geometry. This allows one to extend results like the Bonnet-Meyers theorems to the setting of Finsler geometry. Of course, this then raises questions about what sorts of examples there are of Finsler structures which obey various curvature restrictions and what sorts of topological restrictions are imposed on a manifold by supposing that it carries a Finsler structure satisfying these curvature restrictions.

In particular, for Finsler surfaces, there is an analog of the Gauss curvature, which I continue to call K (but now it is a function on Σ, not M), which governs the stability and focussing of Finsler geodesics. An unexpected result due to Akbar-Zadeh [Ak] asserts that any Finsler structure on a compact surface that satisfies $K \equiv -1$ is a Riemannian metric. He also classifies the Finsler structures on compact surfaces that satisfy $K \equiv 0$, showing that they can only be defined on the torus or Klein bottle and must be so-called 'Minkowski' structures, i.e., quotients of a translation invariant Finsler structure on $\mathbb{R}^2$ by a discrete lattice. This raised the question of whether or not every Finsler structure on S^2 satisfying $K \equiv 1$ must be a Riemannian structure. In §5 of this manuscript, I show that this is not the case, by constructing a family, essentially depending on one arbitrary function of one variable, of non-Riemannian Finsler structures on S^2 that satisfy $K \equiv 1$. In §6 and §4, I show that the hypothesis of compactness in Akbar-Zadeh's results is essential by constructing examples of complete non-Riemannian and non-Minkowskian Finsler structures on $\mathbb{R}^2$ which satisfy $K \equiv -1$ or $K \equiv 0$. This raises interesting questions about the local and global 'generality' of such structures. A significant part of this manuscript is devoted to precisely formulating and answering questions of this type.

I will now give an overview of the contents of the manuscript.

In §1, I review Cartan's construction of a canonical coframing on Σ in the case of a Finsler surface. This coframing is not an arbitrary coframing on a 3-manifold, but satisfies certain structure equations which contain three invariant functions as coefficients: S (which measures 'deviation from a Riemannian structure'), C (which measures the 'deviation from Σ being a circle bundle'), and the aforementioned K.

I also define a generalized Finsler structure to be any 3-manifold endowed with a coframing satisfying these structure equations and explore the relationship between Finsler structures *per se* and generalized Finsler structures. The purpose for considering generalized Finsler structures is to separate essentially 'micro-local' issues having to do with the local generality of coframings satisfying certain differential geometric constraints from local or global issues on the original surface that have to do with the global behavior of certain vector fields or foliations. This separation is an important part of the general approach taken in this manuscript. As an exercise, I use the structure equations to classify the homogeneous Finsler structures on surfaces.

In §2, I derive the Bianchi identities which specify the relations between the covariant derivatives of the three invariants and then use these relations to explore the 'generality' of the generalized Finsler structures which satisfy various curvature conditions, such as requiring S to be constant, requiring C to vanish, requiring K to be constant, etc. All of these conditions have geometric meaning, which I go to some lengths to explain.

The main interesting feature of §2 is the use of Cartan's generalization of Lie's theory of pseudo-groups defined by differential equations to give meaning to such questions as "How many geometrically distinct Finsler structures are there which satisfy $K \equiv 1$?"

An exposition of Cartan's theory would have made the manuscript very long, so I have reluctantly omitted it. The reason for my reluctance is that there is, at present, no modern exposition of this theory. Even Cartan's works only sketch the theory in the intransitive case compared to the level of detail needed. I am finishing a manuscript [Br2] which will include such an exposition and expect that it will be available sometime this fall. The reader who is interested in this aspect of the subject would do well to consult some of Cartan's papers, notably [Ca5], for comparison. That said, I have done my best to keep this aspect of the exposition short and have justified the claims of results by other methods whenever I can.

For example, the method predicts that the generalized Finsler structures satisfying $K \equiv 1$ will depend on two arbitrary functions of two variables modulo diffeomorphism and, in §5, I exhibit a local normal form for such structures which depends on the choice of two arbitrary functions of two variables.

The knowledge of how 'flexible' or 'general' the space of Finsler structures satisfying given curvature conditions is an important guide in working with such structures and in helping one to see what sorts of approaches are liable to work. For example, based on the known generality of generalized Finsler structures satisfying $K \equiv 1$, one should look for a way of describing them by specifying data on a surface, though this surface will not be the original surface M on which the Finsler structure is defined. In §5, I explain how this works. The surface involved is the space of geodesics of the original Finsler structure and working on this surface is the key to constructing a non-Riemannian Finsler structure on S^2 which satisfies $K \equiv 1$.

In §3, I make a short detour into the study of the geodesic flow of a Finsler surface, particularly with an eye towards understanding conditions of a Finsler surface which imply the complete integrability of the geodesic flow. For example, the non-Riemannian Landsberg surfaces are shown to have a completely integrable geodesic flow, as are the generic non-Riemannian surfaces on which K is constant. This brings to mind Cartan's notion [Ca3] of 'geometrically integrable' classes of ordinary differential equations and I remark on this. It seems that there is more to this than meets the eye. I would imagine than a study of the relationship of Finsler geometry with path geometry would clarify this relationship, but I refrained from this in this manuscript because I did not want to lengthen the manuscript with an exposition of path geometry. Still the geometry of the double fibration defined by the geodesic foliation of Σ is intriguing.

In §4, I study the generalized Finsler structures which satisfy $K \equiv 0$. First, I derive a local normal form which depends on two arbitrary functions of two variables, as Cartan's general theory predicts, and then I study the problem of classifying the compact generalized Finsler structures satisfying $K \equiv 0$. My purpose for doing this is mainly didactic. This provides a good example of using the structure equations to derive a global normal form and the results are interesting.

In §5, I study the generalized Finsler structures which satisfy $K \equiv 1$. I begin by showing how the geometry of such structures is encoded in the data of a Riemannian metric and a 1-form defined on the 2-dimensional space of geodesics of the structure. I then use this to construct non-Riemannian Finsler structures satisfying $K \equiv 1$, culminating in a construction of such Finsler structures on S^2. This construction builds on the construction to be found in Darboux [Da] of rotationally invariant Zoll metrics.

Finally, in §6, I briefly redo the analysis, this time to construct complete Finsler structures on $\mathbb{R}^2$ which satisfy $K \equiv -1$. As a bonus, I show how the structure equations can be used to derive Akbar-Zadeh's 'rigidity' result mentioned above for such structures defined on compact surfaces.

1. A Generalization of Finsler Structures on Surfaces

In this section, I will first review basic Finsler geometry and the canonical coframing associated to a Finsler structure on a surface. Then I will consider a generalization which 'micro-localizes' the notion of a Finsler structure, the purpose of which is to allow a separation of local considerations having to do with solving partial differential equations from global considerations having to do with the behavior of the leaf space of certain natural foliations.

1.1. Finsler structures. I will now quickly derive structure equations for Finsler surfaces. This derivation is, of course, not new, having been treated in many places, cf. [BaChSh], [Ca2], [Ch1], [Ga], and [GaWi], to name just a few from different points of view and different eras. The present treatment is mainly to fix notation and nomenclature and treats only the case of surfaces, where there is a notable simplification of the theory. The reader wishing to understand the more general case of Finsler structures on manifolds of higher dimension is urged to consult the above-mentioned references.

Throughout this discussion, M will denote a connected, smooth, oriented surface (i.e., 2-manifold).

Definition 1. A *Finsler structure* on a surface M is a smooth hypersurface $\Sigma^3 \subset TM$ for which the basepoint projection $\pi : \Sigma \to M$ is a surjective submersion having the property that for each $x \in M$, the π-fiber $\Sigma_x = \pi^{-1}(x) = \Sigma \cap T_xM$ is a smooth, closed, strictly convex curve enclosing the origin $0_x \in T_xM$. If, for each $x \in M$, the curve Σ_x is symmetric about 0_x, I say that Σ is *symmetric*.

The prototypical example of a Finsler structure is the unit tangent bundle of a Riemannian metric on M. In fact, fix a Riemannian metric, say g on M and let Σ_1 be its unit tangent bundle. Now choose a smooth positive function r on Σ_1 and define

$$\Sigma_r = \{r(\mathbf{u})^{-1}\,\mathbf{u} \mid \mathbf{u} \in \Sigma_1\}.$$

Then, provided r satisfies a certain natural second order differential inequality (see below), Σ_r will be a Finsler structure on M. Conversely, every Finsler structure on M is of the form Σ_r for some positive function r on Σ_1. Thus, the set of Finsler structures on M can be said to be parametrized by an open set in the space of smooth functions on Σ_1. Informally, one says that Finsler structures 'depend on a choice of one function of three variables'.

Even when Σ is not the unit tangent bundle of a Riemannian metric, it is still possible to regard Σ as the unit circle bundle of a norm $\nu : TM \to \mathbb{R}^+ \cup \{0\}$ that is convex and positively homogeneous of degree 1, i.e., $\nu(\lambda v) = \lambda\,\nu(v)$ for all $\lambda \geq 0$ and $v \in TM$. This makes it possible to use Σ to define the length of an oriented[1] curve $\gamma : [0,1] \to M$ by the formula

$$L_\Sigma(\gamma) = \int_0^1 \nu\big(\gamma'(t)\big)\,dt \tag{1}$$

If γ is an immersion, then there is always an orientation preserving reparametrization which makes γ into a 'unit speed' curve, i.e, one whose velocity vector always lies in Σ. Moreover, the variational problem for L_Σ-length is well-behaved: Through every point of M, in every oriented direction, there will pass a unique, 'unit speed' Σ-geodesic and it will be smooth.

[1]When Σ is symmetric, this notion of length is independent of the orientation.

1.2. The canonical coframing. I am now going to explain how to use the geometry of submanifolds of the tangent bundle to define a natural parallelism on a 3-manifold Σ which defines a Finsler structure on M. This will generalize the construction of a parallelism on the unit circle bundle of a Riemannian metric on a surface, so I will first review this construction.

1.2.1. The Riemannian case. Given a Riemannian metric g on an oriented surface, its unit tangent bundle Σ_1 can be identified with its SO(2)-bundle of oriented orthonormal frames and so carries a canonical parallelization. Recall how this parallelization is defined: For any $\mathbf{u} \in \Sigma_1$, let $J\mathbf{u} \in TM$ denote the unique unit vector with the same basepoint x as $\mathbf{u}$ and having the property that $\bigl(\mathbf{u}, J\mathbf{u}\bigr)$ is an oriented g-orthonormal basis of T_xM. Then there are unique 1-forms α_1 and α_2 on Σ_1 such that, for all $v \in T_{\mathbf{u}}\Sigma_1$, the identity

$$\pi'(\mathbf{u})(v) = \alpha_1(v)\,\mathbf{u} + \alpha_2(v)\,J\mathbf{u}$$

is satisfied. (Recall that $\pi'(\mathbf{u})$ is a surjective linear map from $T_{\mathbf{u}}\Sigma_1$ to T_xM where $x = \pi(\mathbf{u})$, so this makes sense.) These two 1-forms are everywhere linearly independent and there exists a unique 1-form α_3 on Σ_1 satisfying

$$d\alpha_1 = -\alpha_2 \wedge \alpha_3 \qquad \text{and} \qquad d\alpha_2 = -\alpha_3 \wedge \alpha_1\,. \tag{2}$$

This 1-form is linearly independent from the first two and so completes a canonically defined coframing $(\alpha_1, \alpha_2, \alpha_3)$ of Σ_1. Moreover, it satisfies the structure equation

$$d\alpha_3 = -K\,\alpha_1 \wedge \alpha_2 \tag{3}$$

where K represents the Gauss curvature of the Riemannian metric g. In fact, K is constant on each π-fiber and so can be regarded as a function on M.

1.2.2. The general case. The construction of a canonical coframing can be generalized to the Finsler structure case, though the lack of the J operator means that a less direct route has to be taken.

I will now outline this construction. As a check of the various claims that I will make, I will carry the construction out explicitly for a Finsler structure written in the form $\Sigma = \Sigma_r$ where $r > 0$ is a positive function on Σ_1, the unit circle bundle of a fixed Riemannian metric g on M.

I start with a definition. A vector $v \in T_{\mathbf{u}}\Sigma$ will be said to be *monic* if $\pi'(\mathbf{u})(v) = \mathbf{u}$. Since $\pi'(\mathbf{u}) : T_{\mathbf{u}}\Sigma \to T_{\pi(\mathbf{u})}M$ is surjective with a kernel of dimension 1, the set of monic vectors in $T_{\mathbf{u}}\Sigma$ is an affine line. A non-vanishing 1-form θ on Σ will be said to be *null* if $\theta(v) = 0$ for all monic vectors. A 1-form ω will be said to be *monic* if it satisfies $\omega(v) = 1$ for all monic vectors v. Any two null 1-forms are linearly dependent and the difference of any two monic 1-forms is a multiple of a null 1-form.

Let r be the positive function on Σ_1 for which $\Sigma = \Sigma_r$, and let $\rho : \Sigma_r \to \Sigma_1$ be the diffeomorphism which is the inverse of the obvious scaling map, i.e., ρ satisfies

$$\rho\bigl(r(\mathbf{u})^{-1}\,\mathbf{u}\bigr) = \mathbf{u}.$$

Then the 1-form $\bar\theta = \rho^*\alpha_2$ is null, while the 1-form $\bar\omega = \rho^*(r\alpha_1)$ is monic. In particular, null and monic 1-forms exist.

The most general null 1-form is of the form $\theta = \rho^*(t\,\alpha_2)$ and the most general monic 1-form is of the form $\omega = \rho^*\bigl(r\,\alpha_1 + s\,\alpha_2\bigr)$ for some non-zero function t and some function s, both defined on Σ_1.

First, I claim that there is a unique monic 1-form on Σ with the property that its exterior derivative is a multiple of a null 1-form. To show this, I will calculate on Σ_1. Since the α_i form a coframing of Σ_1, there are unique functions r_i on Σ_1 so that

$$dr = r_1\,\alpha_1 + r_2\,\alpha_2 + r_3\,\alpha_3\,. \tag{4}$$

Computation with the structure equations (2–3) yields

$$d\omega\wedge\theta = \rho^*\bigl(d((r\alpha_1 + s\alpha_2)\wedge(t\,\alpha_2))\bigr) = \rho^*\bigl(t(r_3 - s)\,\alpha_1\wedge\alpha_2\wedge\alpha_3\bigr).$$

It follows that

$$\omega = \rho^*\bigl(r\,\alpha_1 + r_3\,\alpha_2\bigr) \tag{5}$$

is the unique monic 1-form on Σ for which $d\omega$ is a multiple of a null 1-form. This canonical 1-form is known in the literature as *Hilbert's invariant form*. I will just call it the Hilbert form. From now on, ω will always denote the Hilbert form.

Note that when $r \equiv 1$, so that $\Sigma = \Sigma_1$ and ρ is the identity map, one has $\omega = \alpha_1$, as might have been expected.

The next canonical 1-form to be defined will be a null 1-form, so I must describe how to choose the function t in a natural way. First, note that the wedge product $\omega\wedge\theta = \rho^*\bigl(rt\,\alpha_1\wedge\alpha_2\bigr)$ of a monic 1-form with a null 1-form is a non-zero multiple of the π-pullback of an area form on M. Using the orientation of M, the sign of t can then be fixed by defining a null 1-form to be *positive* if the wedge of any monic form with it is a positive multiple of the π-pullback of an area form on M. I am now going to look for a canonically defined positive null 1-form.

Now, note that, whatever the value of $t > 0$, the 1-form $\theta = \rho^*(t\,\alpha_2)$ will be a contact 1-form since

$$\theta\wedge d\theta = \rho^*\bigl(t\alpha_2\wedge d(t\alpha_2)\bigr) = \rho^*\bigl(-t^2\,\alpha_1\wedge\alpha_2\wedge\alpha_3\bigr) \neq 0.$$

Now, writing $dr_3 = r_{31}\,\alpha_1 + r_{32}\,\alpha_2 + r_{33}\,\alpha_3$, one computes for the Hilbert form that

$$\omega\wedge d\omega = \rho^*\bigl(-r(r_{33} + r)\,\alpha_1\wedge\alpha_2\wedge\alpha_3\bigr).$$

It follows that ω will also be a contact form and that $\omega\wedge d\omega$ will have the same sign as $\theta\wedge d\theta$ if and only if $r > 0$ satisfies the inequality

$$r_{33} + r > 0. \tag{6}$$

This is the inequality on r to which I alluded earlier. It is not difficult to see that (6) is equivalent to the condition that each π-fiber $\Sigma_x \subset T_xM$ be a strictly convex curve enclosing $0_x \in T_xM$. I will leave the verification of this to the reader. (Note that $r = 1$ satisfies this condition.)

In any case, assuming (6), it follows that there is a unique choice of $t > 0$ so that $\theta\wedge d\theta = \omega\wedge d\omega$. From now on θ will denote the unique positive null 1-form defined by this condition. In terms of forms on Σ_1, the formula for θ is

$$\theta = \rho^*\big(\sqrt{r(r_{33}+r)}\,\alpha_2\big). \tag{7}$$

Note that $\theta = \alpha_2$ when $r \equiv 1$, as expected.

Finally, I describe how to choose a third 1-form. Observe that the 2-forms $d\omega$ and $d\theta$ are nowhere vanishing and that they are also everywhere linearly independent since $\theta\wedge d\theta$ is non-vanishing while $\theta\wedge d\omega$ vanishes identically. It follows that these two 2-forms have a common linear factor, say η, that is unique up to scalar multiples, i.e., η satisfies $\eta\wedge d\omega = \eta\wedge d\theta = 0$. I can then determine η uniquely by requiring, in addition, that $d\omega = \eta\wedge\theta$, so that is what I do.

A straightforward calculation using the formulae derived so far shows that there is a universal cubic polynomial $a(r)$ in r and its α-coframe derivatives up to and including order 3 so that the unique η satisfying these conditions is given by

$$\eta = \rho^*\left(\frac{(r_{33}+r)\,\alpha_3 + \big(r_{31}-r_2\big)\,\alpha_1}{\sqrt{r(r_{33}+r)}} + \frac{a(r)\,\alpha_2}{\sqrt{r^3(r_{33}+r)^3}}\right). \tag{8}$$

(The actual formula for $a(r)$ will not be important in what follows.)

When $r \equiv 1$, one has $\eta = \alpha_3$, as expected. In the general case, I refer to η as the *canonical pseudo-connection.*[2]

The following Proposition asserts the canonical nature of the coframing just constructed. Its proof is implicit in the construction itself, so I will omit it.

Proposition 1. *If $\Sigma \subset TM$ and $\Sigma^* \subset TM^*$ are Finsler structures on the oriented surfaces M and M^* respectively, and $\Phi : M \to M^*$ is an orientation preserving diffeomorphism which satisfies $\Phi'(\Sigma) = \Sigma^*$, then*

$$\big(\Phi'\big)^*\big(\omega^*, \theta^*, \eta^*\big) = \big(\omega, \theta, \eta\big). \quad \square$$

Remark. If Φ reverses orientation and satisfies $\Phi'(\Sigma) = \Sigma^*$, then the pullback equation reads

$$\big(\Phi'\big)^*\big(\omega^*, \theta^*, \eta^*\big) = \big(\omega, -\theta, -\eta\big).$$

By Proposition 1, the group of symmetries of a given Finsler structure is embedded into the group of symmetries of the associated coframing (ω, θ, η) of Σ. In the next section, I will show that these two groups of symmetries are the same. This will allow a classification of the homogeneous Finsler structures analogous to the classification of the homogeneous Riemannian metrics on surfaces.

[2] Since there is no canonical group action on the π-fibers, Σ is not, a priori, a fiber bundle and so η is not actually a connection form in the usual sense, though it does restrict to be a volume form on each π-fiber. For more on this point, see §2.3.

Proposition 2. *For any Finsler structure, the coframing* $\bigl(\omega,\theta,\eta\bigr)$ *satisfies*

$$
\begin{aligned}
d\omega &= \quad \eta\wedge\theta\\
d\theta &= -\eta\wedge\bigl(\omega+S\,\theta\bigr)\\
d\eta &= -\bigl(K\,\omega+C\,\eta\bigr)\wedge\theta
\end{aligned}
\tag{9}
$$

where S*,* K*, and* C *are smooth functions on* Σ*.*

Proof. The equation for $d\omega$ is just the definition. Next, since $\eta\wedge d\theta=0$, it follows that there are functions R and S so that $d\theta=-\eta\wedge\bigl(R\,\omega+S\,\theta\bigr)$. However, the equation $\theta\wedge d\theta=\omega\wedge d\omega$ implies $R\equiv 1$. Finally, since $0=d(d\omega)=d\eta\wedge\theta-\eta\wedge d\theta=d\eta\wedge\theta$, it follows that there are functions K and C so that $d\eta=-\bigl(K\,\omega+C\,\eta\bigr)\wedge\theta$. $\square$

The interpretation of the functions S, C, and K will occupy the next few sections.

1.3. Generalized Finsler structures. With Proposition 2 in hand, I can now define what I mean by a generalized Finsler structure.

Definition 2. A *generalized Finsler structure* on a 3-manifold Σ is a coframing $\bigl(\omega,\theta,\eta\bigr)$ which satisfies the structure equations (9). A generalized Finsler structure will be said to be *amenable* if the leaf space $\mathcal{M}$ of the codimension 2 foliation defined by the equations $\omega=\theta=0$ can be given the structure of a smooth surface M in such a way that the natural projection $\pi:\Sigma\to M$ is a smooth submersion.

Let $\bigl(\mathbf{W},\mathbf{T},\mathbf{H}\bigr)$ be the triple of vector fields on Σ which is dual to the coframing $\bigl(\omega,\theta,\eta\bigr)$. By the usual formulae, the duals of the structure equations are

$$
\begin{aligned}
\bigl[\,\mathbf{H},\,\mathbf{W}\,\bigr] &= \mathbf{T},\\
\bigl[\,\mathbf{H},\,\mathbf{T}\,\bigr] &= -\mathbf{W}+S\,\mathbf{T}+C\,\mathbf{H},\\
\bigl[\,\mathbf{W},\,\mathbf{T}\,\bigr] &= K\,\mathbf{H}.
\end{aligned}
\tag{10}
$$

These equations will be useful below.

Every generalized Finsler structure is locally amenable in the sense that every point of Σ has a neighborhood to which the generalized Finsler structure restricts to be amenable. In fact, the next proposition shows that the difference between the concepts 'Finsler structure' and 'generalized Finsler structure' is global in nature; every generalized Finsler structure is locally diffeomorphic to a Finsler structure.

Proposition 3. *Let* Σ *be a 3-manifold endowed with an amenable generalized Finsler structure* $\bigl(\omega,\theta,\eta\bigr)$*. Denote the* $\omega\theta$*-leaf projection by* $\pi:\Sigma\to M$ *and define a smooth map* $\nu:\Sigma\to TM$ *by the rule* $\nu(\mathbf{u})=\pi'(\mathbf{u})\bigl(\mathbf{W}(\mathbf{u})\bigr)$*. Then* ν *immerses each* π*-fiber* $\Sigma_x=\pi^{-1}(x)$ *as a curve in* T_xM *which is strictly convex towards* 0_x*. Moreover, there is an orientation of* M *so that the* ν*-pullback of the canonical coframing induced on the* ν*-image of* Σ *coïncides with the given generalized Finsler structure.*

Proof. By construction, if $x=\pi(\mathbf{u})$, then $\nu(\mathbf{u})$ is an element of T_xM. Since, by definition, the kernel of π' consists of the tangent vectors $v\in T\Sigma$ which satisfy $\omega(v)=\theta(v)=0$, it follows that $\mathbf{W}(\mathbf{u})$ is not tangent to the fibers of π and hence is not in the kernel of π', so that $\nu(\mathbf{u})\neq 0_x$.

Next, I prove that ν is an immersion. Since $\nu : \Sigma \to TM$ commutes with the basepoint projections and since $\pi : \Sigma \to M$ is a submersion, it suffices to show that the restriction of ν to each π-fiber Σ_x is an immersion. At the same time, I will show that the resulting immersed curve is convex towards 0_x. Let $\mathbf{u} \in \Sigma$ be fixed and set $x = \nu(\mathbf{u})$. Note that the integral curves of $\mathbf{H}$ are the π-fibers. Let $\gamma : (a, b) \to \Sigma$ be the integral curve of $\mathbf{H}$ through $\mathbf{u}$ and let $\bar\gamma = \nu \circ \gamma : (a, b) \to T_xM$. Then (10) implies

$$\bar\gamma'(t) = \pi'\big(\gamma(t)\big)\big(\mathbf{T}(\gamma(t))\big).$$

Since $\mathbf{T}$ and $\mathbf{W}$ span a 2-plane transverse to the fibers of π at every point, it follows that $\bar\gamma(t)$ and $\bar\gamma'(t)$ are linearly independent in T_xM for all $t \in (a, b)$. In particular, ν restricted to each π-fiber is an immersion. Moreover, the equations (10) now further imply

$$\bar\gamma''(t) = -\bar\gamma(t) + S{\circ}\gamma\ \ \bar\gamma'(t). \tag{11}$$

It follows from this that $\bar\gamma$ is strictly convex towards the origin in T_xM.[3]

Thus, the ν-image of Σ is locally identifiable with a Finsler structure as claimed. Since, by definition, each fiber of π is connected, there is a unique orientation of M so that $\omega{\wedge}\theta$ is a positive multiple of the π-pullback of a positive 2-form on M. I will assume that M has been endowed with this canonical orientation.

Use ν locally to identify Σ with its ν-image in TM. Then tracing through the definitions, one sees that ω is monic while θ is a positive null form. Moreover, since $\theta{\wedge}d\omega = 0$, it follows that ω is the Hilbert form. Since $\theta{\wedge}d\theta = \omega{\wedge}d\omega$, it follows that θ is the canonical positive null form. Finally, since $d\omega = \eta{\wedge}\theta$ and $\eta{\wedge}d\theta = 0$, it follows that η is the canonical pseudo-connection. □

A corollary of Proposition 3 is that if $\Sigma \subset TM$ is a Finsler structure, and $\big(\omega, \theta, \eta\big)$ is the associated canonical coframing, then when one regards this coframing as a generalized Finsler structure, one may simply take $\pi : \Sigma \to M$ as the $\omega\theta$-leaf space projection and this yields a mapping ν which is just the identity map.

The reader may well wonder why anyone would bother with generalized Finsler structures since they are locally the same as Finsler structures. The reason is that in the study of Finsler structures defined by geometric conditions, such as conditions on the invariants S, C, and K, one is frequently led to solve differential equations in the larger class of generalized Finsler structures since it is this class that is locally defined. Then, as a separate step, one can determine the necessary conditions on a generalized Finsler structure that it actually be a Finsler structure. Thus, generalized Finsler structures provide a natural intermediate stage where problems can be localized and solved without the complication of global issues. I will give several examples of this strategy in the sections to follow.

There is a simple necessary and sufficient test for a generalized Finsler structure to be a Finsler structure.

[3] If V is an oriented 2-dimensional vector space, a smooth curve $\phi : (a, b) \to V$ that satisfies $\phi{\wedge}\phi' \neq 0$ has a natural orientation so that $\phi{\wedge}\phi' > 0$. If, in addition, ϕ satisfies the open condition $\phi'{\wedge}\phi'' > 0$, then locally its image is strictly convex towards the origin. In this case, there is a unique element of arc, called the *centro-affine element*, so that $\phi{\wedge}\phi' = \phi'{\wedge}\phi'' > 0$. Using this element, $\phi'' = -\phi + s\,\phi'$ where $s : (a, b) \to \mathbb{R}$ is the so-called *centro-affine curvature*. It follows from (11) that dt is the natural centro-affine element for $\bar\gamma$ and that $S{\circ}\gamma$ is its centro-affine curvature, see [GaWi].

Proposition 4. *A generalized Finsler structure* (ω, θ, η) *on a 3-manifold* Σ *is a Finsler structure if and only if its* $\omega\theta$*-leaves are compact, it is amenable, and the canonical immersion* $\nu : \Sigma \to TM$ *is one-to-one on each* π*-fiber.*

The proof is straightforward. Note, however, that just having the $\omega\theta$-leaves be compact does not make the structure amenable since one could have a discrete subset of the leaves around which the foliation is not locally a product, the so-called 'ramified' circles. In this case, the leaf space will have the structure of a 2-dimensional orbifold near the ramified points. Moreover, even if the structure is amenable, the ν-image of each π-fiber Σ_x will in general be a closed, strictly convex curve in T_xM which winds around the origin μ times for some positive integer μ. This number μ, the *multiplicity* of the generalized Finsler structure, is equal to 1 if and only if each Σ_x is embedded via ν. Both sorts of phenomena can occur in practice.

Next, I state a local structure theorem for generalized Finsler structures.

Proposition 5. *Let* (ω, θ, η) *be a generalized Finsler structure on a 3-manifold* Σ*. Then every* $\mathbf{u} \in \Sigma$ *has an open neighborhood* U *on which there exists a* $\mathbf{u}$*-centered coordinate chart* $(x, y, p) : U \to D \subset \mathbb{R}^3$ *and a function* $L : U \to \mathbb{R}$ *satisfying* $L, L_{pp} > 0$ *so that, on* U*, the following formulae hold:*

$$
\begin{aligned}
\omega &= L\,dx + L_p\,(dy - p\,dx)\\
\theta &= \sqrt{L\,L_{pp}}\,\bigl(dy - p\,dx\bigr)\\
\eta &= \frac{dL_p - L_y\,dx}{\sqrt{L\,L_{pp}}} + \frac{A(L)\,(dy - p\,dx)}{\sqrt{(L\,L_{pp})^3}}
\end{aligned}
\tag{12}
$$

where $A(L)$ *is a certain universal polynomial in* L *and its derivatives that is determined by the identity* $\eta\wedge d\theta = 0$*.*

Conversely, if L *is a positive function on a domain* D *in* xyp*-space and satisfies* $L_{pp} > 0$ *on* D*, then the formulae* (12) *define a generalized Finsler structure on* D*.*

Informally, Proposition 5 states that the local generalized Finsler structures depend on one function of three variables. I will refer to coordinates satisfying the conditions of Proposition 5 as *contact coordinates* and the function L will be called the *Lagrangian* of the structure in the given contact coordinates. The reason for this terminology is classical and will be further explored in §3.

Proof. I will only sketch the proof. Let x and y be functions on a neighborhood of u so that $\omega\wedge\theta = F\,dx\wedge dy$ for some function $F > 0$. (Geometrically, x and y are just independent local functions whose level curves define the $\omega\theta$-foliation.) By switching x and y if necessary, I can assume that $\theta\wedge dx$ is non-zero at $\mathbf{u}$ and by replacing x and y by $x - x(\mathbf{u})$ and $y - y(\mathbf{u})$, I can assume that $x(\mathbf{u}) = y(\mathbf{u}) = 0$. Since θ is not a multiple of dx near $\mathbf{u}$, I know that there is a positive function H and another function p, both defined on a neighborhood of $\mathbf{u}$, so that that

$$\theta = H\,(dy - p\,dx).$$

Since $0 \neq \theta\wedge d\theta = -H^2\,dx\wedge dy\wedge dp$, it follows that x, y and p form a coordinate system near $\mathbf{u}$. Replacing y and p by $y - p(\mathbf{u})\,x$ and $p - p(\mathbf{u})$ respectively, I can assume that $p(\mathbf{u}) = 0$. Since $\omega{\wedge}\theta \neq 0$, it follows that there exist functions $L > 0$ and M, so that

$$\omega = L\,dx + M\,(dy - p\,dx).$$

The condition $\theta{\wedge}d\omega = 0$ implies $M = L_p$. Then the equation $\theta{\wedge}d\theta = \omega{\wedge}d\omega \neq 0$, implies that $H^2 = L\,L_{pp}$. The rest of the normalizations can now be completed by the reader. $\square$

The invariants S, C, and K can be computed in terms of L. For example, using the formulae (12), to expand $d(\omega{\wedge}\theta) = S\,\omega{\wedge}d\omega$ yields

$$S = -\frac{L\,L_{ppp} + 3L_p\,L_{pp}}{2\,\sqrt{(L\,L_{pp})^3}} = -\frac{(L^2)_{ppp}}{4\,\sqrt{(L\,L_{pp})^3}}\,.$$

In consequence, $S = 0$ if and only if L^2 is quadratic in p. (Since I am assuming that $L\,L_{pp} > 0$, it follows that $\frac{1}{2}\bigl(L^2\bigr)_{pp} = L\,L_{pp} + \bigl(L_p\bigr)^2 > 0$, so L^2 cannot be linear in p. In fact, the hypothesis $L\,L_{pp} > 0$ together with $S = 0$ implies that L^2 is a strictly positive quadratic form in p, so that L is the arc length Lagrangian for a Riemannian metric on the $\omega\theta$-leaf space, see §2.2.) The formulae for C and K in terms of L are more complicated and will not be discussed or needed.

1.4. Symmetries and homogeneous examples. In the study of any geometry structure, the homogeneous examples and, more generally, the symmetries of a given example play an important role. In this section, I will discuss the group of symmetries of (generalized) Finsler structures and the 'generality' of homogeneous examples.

1.4.1. The group of symmetries. First, a pair of definitions.

Definition 3. Let $\Sigma \subset TM$ be a Finsler structure on M. A *symmetry* of Σ is a diffeomorphism $\Phi : M \to M$ which satisfies $\Phi'\bigl(\Sigma\bigr) = \Sigma$. Let $\bigl(\omega, \theta, \eta\bigr)$ be a generalised Finsler structure on a 3-manifold P. A *symmetry* of $\bigl(\omega, \theta, \eta\bigr)$ is a diffeomorphism $\Psi : P \to P$ which satisfies $\bigl(P^*\omega,\, P^*\theta,\, P^*\eta\bigr) = \bigl(\omega, \theta, \eta\bigr)$.

Using this terminology, I can combine Propositions 1 and 3 into the following result, whose proof is simple enough that I can omit it.

Proposition 6. *If $\Sigma \subset TM$ is a Finsler structure on M, then the assignment $\Phi \mapsto \Phi'$ gives a one-to-one correspondence between the orientation preserving symmetries of the Finsler structure on M and the symmetries of the canonical generalized Finsler structure on Σ.* $\square$

Note that one can always reduce to the orientation preserving case by passing to the orientation double cover $\tilde{M}$ of M and considering the lifted Finsler structure $\tilde{\Sigma} \subset T\tilde{M}$.

A corollary of Proposition 6 is that the group of symmetries of a Finsler structure can be given the structure of a Lie group of dimension at most 3. This follows from a theorem of Kobayashi [Ko], which asserts that, for any connected manifold P endowed with a coframing α, the group $G \subset \mathrm{Diff}(P)$ of symmetries of α can be given a Lie group structure in a unique way so that, for each $p \in P$, the evaluation map $\mathrm{ev}_p : G \to P$, defined by $\mathrm{ev}_p(\gamma) = \gamma(p)$ for all $\gamma \in G$, be a smooth embedding of G as a closed submanifold of P.

There are two senses in which a Finsler structure could be said to be homogeneous. First, one could say that $\Sigma \subset TM$ is homogeneous if its group of symmetries acts transitively on M. A more restrictive notion of homogeneity would be to require that the group of symmetries act transitively on Σ itself. Both notions are interesting and will be discussed in the remainder of this section.

1.4.2. Symmetry groups of dimension 3. Note that any γ in the group G of symmetries of a generalized Finsler structure $\big(\omega, \theta, \eta\big)$ on P must satisfy $\gamma^*S = S$, $\gamma^*C = C$, and $\gamma^*K = K$. In particular, each of these three functions must be constant on the G-orbits in P. Since the G-orbits are automatically closed in P, if the group of symmetries of a (generalized) Finsler structure on a connected manifold is to have the maximum possible dimension of 3, then each of the functions S, C, and K must be constant.

If one imposes the conditions $dS = dC = dK = 0$ and differentiates the equations (9), the result is, first, that $0 = d\big(d\theta\big) = -C\,\omega\wedge\theta\wedge\eta$, implying that $C = 0$, and, second, that $0 = d\big(d\eta\big) = KS\,\omega\wedge\theta\wedge\eta$, implying that $KS = 0$. Thus, there are two possible types of homogeneous generalized Finsler structures, those which satisfy $S = C = 0$ and those which satisfy $K = C = 0$.

The first type is easily identified. These are essentially the Finsler structures which arise from homogeneous Riemannian metrics on surfaces. In fact, for each real number K, consider the connected Lie subgroup $G_K \subset \mathrm{GL}(3,\mathbb{R})$ whose Lie algebra $\mathfrak{g}_K \subset \mathfrak{gl}(3,\mathbb{R})$ consists of matrices of the form

$$\begin{pmatrix} 0 & -Kx & -Ky \\ x & 0 & -z \\ y & z & 0 \end{pmatrix}.$$

for x, y, and z in $\mathbb{R}$. Let $H \subset G_K$ be the connected subgroup whose Lie algebra consists of matrices of the above form with $x = y = 0$. Then $H \simeq \mathrm{SO}(2)$ and G_K acts on the simply connected homogeneous space $M_K = G_K/H$ as the group of oriented isometries of a Riemannian metric on M_K of constant Gaussian curvature K. In fact, G_K can be identified as the unit tangent bundle of M_K endowed with this metric.

Given a homogeneous generalized Finsler structure $\big(\omega, \theta, \eta\big)$ satisfying $S = C = 0$ on a simply connected 3-manifold P, the $\mathfrak{g}_K$-valued 1-form ϕ defined by

$$\phi = \begin{pmatrix} 0 & -K\omega & -K\theta \\ \omega & 0 & -\eta \\ \theta & \eta & 0 \end{pmatrix}$$

satisfies the structure equation $d\phi = -\phi\wedge\phi$, so there is a covering map $g : P \to G_K$ satisfying $\phi = g^{-1}\,dg$.

The second type is also easily identified. For each fixed constant S, these are essentially the generalized Finsler structures associated to the flat H_S-structures on surfaces where $H_S \subset \mathrm{GL}(2,\mathbb{R})$ is the connected 1-parameter subgroup whose Lie algebra is

$$\mathfrak{h}_S = \left\{ \begin{pmatrix} Sz & z \\ -z & 0 \end{pmatrix} \,\middle|\, z \in \mathbb{R} \right\}.$$

In fact, given such a homogeneous generalized Finsler structure on a simply connected 3-manifold P satisfying $K = C = 0$, if one sets

$$\phi = \begin{pmatrix} 0 & 0 & 0 \\ \omega & S\eta & \eta \\ \theta & -\eta & 0 \end{pmatrix}$$

then the structure equations (9) can be written in the form $d\phi = -\phi{\wedge}\phi$. It follows that there is a map $g : P \to \mathrm{GL}(3, \mathbb{R})$ that satisfies $\phi = g^{-1}\,dg$ and that, moreover, this map is a covering map onto a certain 3-dimensional subgroup.

1.4.3. Symmetry groups of dimension 2. More interesting is the case of Finsler structures $\Sigma \subset TM$ for which the group G of symmetries acts transitively on M but not on Σ. In this case, the group G will have to be of dimension 2 and its action on M will have a finite stabilizer subgroup. By passing to the simply connected cover of M, I can assume that the identity component G° of G acts simply transitively on M. Thus, for any $m \in M$, the evaluation map $\mathrm{ev}_m : G^\circ \to M$ is a diffeomorphism. For this reason, I might as well fix an element of m and identify M with G° via ev_m. Under this identification, the action of G° on M simply becomes the left-action of G° on itself via Lie group multiplication.

Since Σ is invariant under G°, it follows that Σ_e determines Σ_g for all other $g \in G^\circ$. One simply has $\Sigma_g = L_g'(\Sigma_e)$ where $L_g : G^\circ \to G^\circ$ is left multiplication by g. Conversely, starting with any closed curve C in $T_eG^\circ \simeq \mathbb{R}^2$ which is strictly convex towards the origin, there will be a unique left-invariant Finsler structure $\Sigma \subset TG^\circ$ which satisfies $\Sigma_e = C$.

Thus, the homogeneous Finsler structures of this type are generated by choosing a 1-connected Lie group G° of dimension 2 and a closed curve C in T_eG° which is strictly convex towards the origin.

Up to isomorphism, there are two 1-connected Lie groups of dimension 2. The first is the abelian group $\mathbb{R}^2$ and the second is the (solvable) non-abelian group $N \subset \mathrm{GL}(2, \mathbb{R})$ of matrices of the form

$$\begin{pmatrix} e^y & x \\ 0 & 1 \end{pmatrix}$$

where x and y are real.

In the case where G° is the abelian $\mathbb{R}^2$, the resulting structures are known as the *Minkowski* structures. The structures whose underlying symmetry group is the non-abelian group N do not seem to have a special name in the literature.

2. Special Classes of Finsler Structures

In this section, I want to consider some special classes of Finsler structures defined by various curvature conditions. The objective of this section is to discuss the 'generality' of the Finsler structures satisfying given conditions where 'generality' is meant in the sense that Cartan describes, for example, in [Ca4] or [Ca5]: Since the conditions are invariant under the diffeomorphism pseudo-group in three dimensions, in a crude sense the space of Finsler structures satisfying some given geometric condition depends on three arbitrary functions of three variables (if it is not empty). However, one wants to give a sense to how many functions of how many variables are needed to describe the local *equivalence classes* of solutions, where 'equivalence' means 'diffeomorphic'.

Cartan gave a sense to this in the analytic category via his generalization of the fundamental existence and uniqueness theorems of Lie concerning local Lie groups to existence and uniqueness theorems for coframings[4] subject to differential conditions. There is no space here to recount this theory, but, fortunately, it is frequently possible to avoid its use, as I shall do in much of this section. However, many of the ideas presented here are directly motivated by that theory, so the reader may want to consult an account of it. While a modern source suitable for this manuscript is being prepared [Br2] and modern examples exist [Br1], there is no substitute for the original source and [Ca5], especially Part III, is a good place to start.

2.1. The Bianchi identities. The structure equations of a generalized Finsler structure (ω, θ, η) on a 3-manifold Σ are

$$\begin{aligned} d\omega &= \quad \eta \wedge \theta \\ d\theta &= -\eta \wedge \bigl(\omega + S\,\theta\bigr) \\ d\eta &= -\bigl(K\,\omega + C\,\eta\bigr) \wedge \theta \end{aligned} \tag{1}$$

where S, K, and C are smooth functions on Σ. Computing the exterior derivative of the second equation[5] yields

$$\begin{aligned} 0 = d\bigl(d\theta\bigr) &= -d\eta \wedge \bigl(\omega + S\,\theta\bigr) + \eta \wedge \bigl(d\omega + dS \wedge \theta + S\,d\theta\bigr) \\ &= \bigl(K\,\omega + C\,\eta\bigr) \wedge \theta \wedge \bigl(\omega + S\,\theta\bigr) + \eta \wedge \bigl(\eta \wedge \theta + dS \wedge \theta - S\,\eta \wedge \bigl(\omega + S\,\theta\bigr)\bigr) \\ &= \bigl(dS - C\,\omega\bigr) \wedge \theta \wedge \eta, \end{aligned}$$

which implies that there exist functions S_2 and S_3 so that

$$dS = C\,\omega + S_2\,\theta + S_3\,\eta. \tag{2a}$$

The derivative of the third equation yields, after some simplification and arrangement,

$$0 = -\bigl(dK \wedge \omega + (dC + KS\,\omega) \wedge \eta\bigr) \wedge \theta,$$

which implies that there exist functions K_1, K_2, K_3, C_2, and C_3 so that

$$\begin{aligned} dK &= \qquad\qquad K_1\,\omega + K_2\,\theta + K_3\,\eta \\ dC &= (K_3 - KS)\,\omega + C_2\,\theta + C_3\,\eta \end{aligned} \tag{2 b,c}$$

The formulae $(2a{-}c)$ constitute the Bianchi identities of the structure.

The tableau of the free derivatives of the three invariants has characters $s_1 = 3$, $s_2 = 3$, and $s_3 = 1$; is involutive; and has a trivial torsion cokernel. Thus, the

[4] I.e., $\{e\}$-structures.

[5] The derivative of the first equation is an identity.

general theory of Cartan would say that the general coframing satisfying these equations depends on one function of three variables up to diffeomorphism, which agrees with the observation that the hypersurfaces in the tangent bundle of a surface depend on one function of three variables as well as with Proposition 5, which gives a local normal form for such coframings in terms of one function of three variables.

I now want to investigate various special conditions that can be put on generalized Finsler structures on surfaces and look at the generality of the resulting systems.

2.2. The condition $S = 0$. First, consider the case $S = 0$. By the Bianchi identities above, this forces $C = 0$, which, in turn forces $K_3 - KS = K_3 = 0$. Thus, the structure equations simplify to

$$\begin{aligned} d\theta &= -\eta \wedge \omega, \\ d\omega &= \quad \eta \wedge \theta, \\ d\eta &= -K\,\omega \wedge \theta, \\ dK &= K_1\,\omega + K_2\,\theta. \end{aligned}$$

The tableau of the free derivatives has characters $s_1 = 1$, $s_2 = 1$, and $s_3 = 0$; is involutive; and has vanishing torsion cokernel. By the general theory of Cartan, up to diffeomorphism these coframings depend on one function of two variables.

It is easy to make this dependence explicit. The structure equations imply that the Lie derivative of the quadratic form $g = \theta^2 + \omega^2$ with respect to the vector field $\mathbf{H}$ vanishes, so that g is a well-defined Riemannian metric on any local leaf space of the line field $\omega = \theta = 0$. Moreover, the structure equations show that the canonical local embedding of Σ into these leaf spaces reveals it as an open subset of the unit sphere bundle of g, so that the generalized Finsler structure is locally a Riemannian structure.

Of course, this statement also follows from the formula for S in canonical coordinates given in §1.3. There, it was shown that S vanishes if and only if L^2 is quadratic in p, which implies that any ν-image of Σ is the set of unit vectors of a positive definite quadratic form on the leaf space.

Conversely, the formulae of §1.2 show that the canonical Finsler structure of a Riemannian metric on a surface satisfies $S = 0$ since in that case, one can identify Σ with Σ_1 for the given Riemannian metric.

Thus $S = 0$ is the condition that a generalized Finsler structure be locally Riemannian. As is well-known, up to diffeomorphism, the Riemannian metrics on surfaces depend on one function of two variables, confirming the count of the general theory. Explicitly, corresponding to isothermal coordinates for the metric g, there will exist canonical coordinates in which the Lagrangian L has the form

$$L = e^{u(x,y)}\sqrt{1+p^2}$$

where u is an arbitrary function of x and y.

In this case, the invariant K is well-defined on the local leaf spaces and is just the Gauss curvature of the metric g. The general theory of differential invariants of a coframing now specialize in this case to the statement that the differential invariants of a Riemannian metric on a surface are all formed from the covariant derivatives of the Gauss curvature.

A more general condition would be to require that S be a constant, not necessarily zero. Of course, this also forces $C = 0$ and thence $K_3 = KS$. As the reader can check, this system, too, is involutive, with the general coframing depending on one function of two variables, just as before. In this case the reader can verify that the generalized Finsler structure defines an H_S-structure on any local leaf space of $\theta = \omega = 0$, where

$$\mathfrak{h}_S = \left\{ \begin{pmatrix} Sx & x \\ -x & 0 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}$$

For $S \neq 0$, this is a non-compact one-parameter subgroup of $\mathrm{GL}(2, \mathbb{R})$. For example, when $S^2 < 4$, the canonical local embeddings realize the fibers of π as 'logarithmic spirals' in the tangent spaces. Note that, for $S \neq 0$, none of these generalized Finsler structures is actually realizable as a Finsler structure on a surface.

2.3. Landsberg surfaces. A generalized Finsler structure is said to be a *Landsberg* structure if it satisfies the condition $C = 0$. Then $dC = 0$ forces $K_3 = KS$, and the remaining Bianchi identities on the invariants become

$$\begin{aligned} dS &= \qquad\quad + S_2\,\theta + \ \ S_3\,\eta \\ dK &= K_1\,\omega + K_2\,\theta + KS\,\eta \end{aligned}$$

The tableau of the free derivatives has characters $s_1 = 2$, $s_2 = 2$, $s_3 = 0$; is involutive; and has vanishing torsion cokernel. Thus, the general theory of Cartan yields that, modulo diffeomorphism, such structures depend on two functions of two variables.

The equation for a covector $\xi = \xi_1\,\omega + \xi_2\,\theta + \xi_3\,\eta$ to be characteristic is that $\xi_1\xi_3 = 0$. From the general theory, this has the following consequence: Suppose that two real-analytic Landsberg structures (ω, θ, η) and $(\tilde{\omega}, \tilde{\theta}, \tilde{\eta})$ on a real analytic 3-manifold Σ satisfy the equations

$$\omega = \tilde{\omega}, \qquad \theta = \tilde{\theta}, \qquad \eta = \tilde{\eta}, \qquad S = \tilde{S}, \qquad K = \tilde{K},$$

along a real analytic surface $P \subset \Sigma$ which satisfies the non-degeneracy condition that neither of the 2-forms $\omega{\wedge}\theta$ nor $\theta{\wedge}\eta$ vanish on P. Then there is a real analytic map $\phi : U \to \Sigma$ of an open neighborhood U of P that fixes P, is a local diffeomorphism, and identifies the two structures.

In this case, it is not so easy to write down an explicit normal form which depends on two arbitrary functions of two variables. However, special cases can easily be integrated. For example, the case $S = 0$ has already been considered and the case $K = 0$ will be considered in §4, where it will be shown that the structures satisfying $K = C = 0$ depend on one arbitrary function of two variables and an explicit normal form will be constructed.

One of the reasons for interest in Landsberg structures is the following characteristic geometric property:

Proposition. *For a Finsler structure $\Sigma \subset TM$, the condition $C = 0$ is the necessary and sufficient condition that there be a free* $\mathrm{SO}(2)$*-action on Σ whose orbits are the π-fibers and that leaves η invariant. In other words, the condition $C = 0$ is equivalent to the condition that Σ can be given the structure of a principal right* $\mathrm{SO}(2)$*-bundle in such a way that η defines a connection on Σ.*

Proof. First, suppose that $C = 0$. Consider the function ψ on M defined by

$$\psi(x) = \int_{\Sigma_x} \eta,$$

where the integral is taken in the positive sense around the circle Σ_x (i.e., the orientation for which η is positive). Now $\psi > 0$ on M. For any two points x and y in M which are in the same connected component, let $\gamma \subset M$ be a smooth, embedded, oriented curve joining x to y. Then $\Sigma_\gamma = \pi^{-1}(\gamma) \subset \Sigma$ is a smooth embedded cylinder on which $\omega \wedge \theta$ vanishes identically. For the obvious orientation on Σ_γ, one has the formula

$$\psi(y) - \psi(x) = \int_{\Sigma_y} \eta - \int_{\Sigma_x} \eta = \int_{\Sigma_\gamma} d\eta = \int_{\Sigma_\gamma} -K\,\omega \wedge \theta = 0,$$

so that ψ is locally constant. Let $\mathbf{Z} = (2\pi/\psi)\mathbf{H}$. Note that the flow of $\mathbf{Z}$ is periodic of period 2π. Thus, there exists an action of SO(2) on Σ so that its induced vector field is $\mathbf{Z}$. Moreover, due to the local constancy of ψ, a computation from the structure equations gives

$$\mathcal{L}_{\mathbf{Z}}\,\eta = d\big(\eta(\mathbf{Z})\big) + \mathbf{Z} \lrcorner\, d\eta = d\big(2\pi/\psi\big) + Z \lrcorner \big(-K\,\omega \wedge \theta\big) = 0,$$

so that η is invariant under the flow of $\mathbf{Z}$ and hence is indeed a connection on Σ with respect to this SO(2)-bundle structure.

Conversely, suppose that there is a free action of SO(2) on Σ whose orbits are the π-fibers. Let $\mathbf{Z}$ denote the vector field which generates this action and whose flow has period 2π. Then $\omega(Z) = \theta(Z) = 0$ and, by reversing the sign of $\mathbf{Z}$ if necessary, I can suppose that $\eta(Z) = f > 0$ for some function f on Σ. Now, if η is to be a connection on Σ with respect to this action then it must be invariant under the flow of $\mathbf{Z}$, i.e., $\mathcal{L}_{\mathbf{Z}}\,\eta = 0$. This condition expands, via the structure equations, to

$$0 = \mathcal{L}_{\mathbf{Z}}\,\eta = d\big(\eta(\mathbf{Z})\big) + \mathbf{Z} \lrcorner\, d\eta = df - C\,\theta.$$

However, $C\,\theta = df$ implies $C^2\theta{\wedge}d\theta = (C\,\theta){\wedge}d(C\,\theta) = df{\wedge}0 = 0$. Since $\theta{\wedge}d\theta \neq 0$, it follows that C must vanish identically. □

Remark. In the case where $C = 0$ and M is connected, the function ψ is actually constant, so that $\hat\eta = (2\pi/\psi)\,\eta$ is the natural connection form on Σ. It follows that there is a 2-form Ω on M so that $d\hat\eta = -\pi^*(\Omega)$. This 2-form is the curvature 2-form of the connection $\hat\eta$. If M is compact, a simple calculation gives[6]

$$\int_\Sigma \eta \wedge d\eta = \left(\frac{\psi}{2\pi}\right)^2 \int_\Sigma \hat\eta \wedge d\hat\eta = -\frac{\psi^2}{2\pi} \int_M \Omega = -\psi^2\,\chi(\Sigma) = -\psi^2\,(2 - 2g)$$

(where g is the genus of M), since Σ is homotopic as a bundle to the unit circle bundle of any Riemannian structure on M. This is the appropriate version of the Gauss-Bonnet theorem for Landsberg Finsler structures on surfaces.

[6]The orientation of Σ is the one for which $\omega{\wedge}\theta{\wedge}\eta$ is a positive 3-form.

2.4. Berwald structures. A structure is said to be *Berwald* if it satisfies $dS \equiv 0 \mod \eta$. In particular, this implies that C vanishes, so that such structures are Landsberg.

On the open set where $dS \neq 0$, say, $dS = S_2\,\eta$ where $S_2 \neq 0$, it follows that $\eta = dS/S_2$ must be integrable, so that $0 = \eta{\wedge}d\eta = -K\,\omega{\wedge}\theta{\wedge}\eta$. In other words, K must be zero. The structures satisfying $K = C = 0$ and $dS{\wedge}\eta = 0$ are known as the *Minkowski* structures. There is a simple local normal form for them, depending on one arbitrary function of one variable. It can be derived as follows:

Since $d\eta = 0$, it can be written locally in the form $\eta = dp$. More precisely, every point in Σ has a 1-connected neighborhood U on which one can choose a function p, unique up to an additive constant, so that $\eta = dp$. For convenience, I will assume that this neighborhood has the property that the level sets of p are connected. Moreover, by shrinking U if necessary, I can arrange that the integral curves of the vector field H in U are all connected as well. Since $dS{\wedge}dp = 0$, it follows that S can be written on U as a function of p, say $S = s(p)$. Then there exists a $\mathrm{GL}(2,\mathbb{R})$ valued function A on the interval $p(U) \subset \mathbb{R}$ so that

$$dA = A\begin{pmatrix} 0 & -\,dp \\ dp & s(p)\,dp \end{pmatrix} = A\begin{pmatrix} 0 & -\,\eta \\ \eta & S\,\eta \end{pmatrix}$$

and A is unique up to left multiplication by a constant matrix. Now a computation from the structure equations gives

$$d\left(A\begin{pmatrix}\omega \\ \theta\end{pmatrix}\right) = 0.$$

Thus, it follows that there exist functions x and y so that

$$\begin{pmatrix}\omega \\ \theta\end{pmatrix} = A^{-1}\begin{pmatrix}dx \\ dy\end{pmatrix}$$

Note that these structures are translation invariant in x and y, which are, of course, coordinates on the $\omega\theta$-leaf space. In this sense, the corresponding structures are 'flat'.

In case the set where dS vanishes has any interior, say V, then the structure is locally equivalent that of an H_S-structure, as discussed in §2.2. As already remarked there, unless $S = 0$, these generalized Finsler structures do not arise from Finsler structures.

2.5 K-basic structures. A more general class of Finsler structures than the ones coming from Riemannian geometry are the ones for which K is well-defined on the base manifold M. For generalized Finsler structures, this is equivalent to the assumption that K be constant along the $\omega\theta$-leaves, i.e., that $K_3 = 0$. I will call these structures *K-basic*.

For such structures, the tableau of the free derivatives has $s_1 = 3$, $s_2 = 3$, $s_3 = 0$; is involutive; and has a vanishing torsion cokernel. It follows that, up to diffeomorphism, these structures depend on three functions of two variables. The equation for a covector $\xi = \xi_1\,\omega + \xi_2\,\theta + \xi_3\,\eta$ to be characteristic is $\xi_3(\xi_1)^2 = 0$.

More restrictive than this is the class of structures for which K is actually constant. For such structures, the tableau of the free derivatives has $s_1 = 2$, $s_2 = 2$, and $s_3 = 0$; is involutive; and has a vanishing torsion cokernel. It follows that, up to diffeomorphism, these structures depend on two functions of two variables. Local normal forms for these structures will be constructed in the sections below. The equation for a covector $\xi = \xi_1\,\omega + \xi_2\,\theta + \xi_3\,\eta$ to be characteristic is $(\xi_1)^2 = 0$.

The interest in these structures arises from certain facts, to be discussed in the next section, about the geodesic flow for such structures.

3. Integrability of the Geodesic Flow

One of the reasons Finsler structures are interesting is that they globalize the classical first order calculus of variations problem for one function of one variable. The reader will recall that this is the problem of extremizing the integral

$$I(u) = \int_a^b L\big(x, u(x), u'(x)\big)\,dx$$

where $L(x, y, p)$ is a given function of three variables.

When the function L is not zero, this translates into a (generalized) Finsler structure as follows: Let $\Sigma \subset \mathbb{R}^3$ be the open subset of the domain of the function L defined by $L \neq 0$. Then Σ can be immersed into $T\mathbb{R}^2$ as the set of vectors of the form

$$\frac{1}{L(x,y,p)}\left(\frac{\partial}{\partial x} + p\,\frac{\partial}{\partial y}\right),$$

so that Σ (at least locally) defines a Finsler structure on a domain in $\mathbb{R}^2$ whenever the Lagrangian L is positive and convex in p.

In this case, the 1-form $L\,dx$ is monic and the 1-form $dy - p\,dx$ is null. The assumption $L\,L_{pp} > 0$ then is equivalent to the local convexity condition and so a generalized Finsler structure can be defined, resulting in the coframing on Σ given by the formulae of Proposition 5 of §1.3.

Under suitable smoothness hypotheses, one derives the Euler-Lagrange criterion, which says that a function u of x renders I extremal if and only if it satisfies the second order ODE

$$\frac{d}{dx}\left(L_p\big(x, u(x), u'(x)\big)\right) - L_y\big(x, u(x), u'(x)\big) = 0.$$

In other words, u renders the integral I extremal if and only if the 1-graph $(x, y, p) = \big(t, u(t), u'(t)\big)$ is an integral curve of the system

$$dy - p\,dx = dL_p - L_y\,dx = 0.$$

By the formulae in Proposition 5, this can be expressed in terms of the generalized Finsler structure as saying that the 1-graphs of the solutions to the Euler-Lagrange equations are the integral curves of the system $\theta = \eta = 0$.

In the case of Riemannian metrics, the extremals of the length functional are called geodesics, which I use to motivate the following definitions:

Definition. Let (ω,θ,η) be a generalized Finsler structure on a 3-manifold Σ. The *geodesics* of the structure are its $\theta\eta$-integral curves. They define a foliation of Σ called the *geodesic foliation*. The (local) flow of $\mathbf{W}$ (the dual of ω) is the *geodesic flow* and a generalized Finsler structure is *geodesically complete* if $\mathbf{W}$ is complete on Σ, i.e., its flow exists for all time. The structure is *geodesically amenable* if the leaf space Λ of the geodesic foliation can be given the structure of a smooth surface in such a way that the natural projection $\ell:\Sigma\to\Lambda$ is a smooth submersion.[7]

When $\Sigma\subset TM$ is a Finsler structure on a surface M, a 'Σ-geodesic' will mean an immersed curve $\gamma: I\to M$ (where I is an interval in $\mathbb{R}$) that satisfies the condition that, first, $\gamma'(t)$ lie in Σ for all $t\in I$ and, second, that the lifted curve $\gamma': I\to\Sigma$ be a geodesic of the generalized Finsler structure on Σ.

If $\gamma: I\to M$ is a Σ-geodesic and the interval I contains 0, then $\gamma'(t)=\exp_{t\mathbf{W}}(\mathbf{u})$ where $\mathbf{u}=\gamma'(0)$. Thus, $\gamma(t)=\pi\bigl(\exp_{t\mathbf{W}}(\mathbf{u})\bigr)$.

3.1. The meaning of the invariant K. In the Riemannian geometry of surfaces, the Gaussian curvature plays an important role in the formula for the second variation of arc length. In the more general case of a Finsler structure, the function K plays the same role. More precisely, a compact Σ-geodesic $\gamma:[a,b]\to M$ will be a local minimum of the Σ-length functional L_Σ defined in §1.1 on curves joining $\gamma(a)$ to $\gamma(b)$ if the quadratic form

$$Q_\gamma(f)=\int_a^b\bigl((f')^2-K_\gamma\,f^2\bigr)\,dt$$

has zero index and nullity on the space of smooth functions on $[a,b]$ which vanish at the endpoints where $K_\gamma=K\circ\gamma'$. This follows by an elementary calculation which I will omit.

In particular, note that if K is non-positive, then every geodesic segment is locally minimizing. On the other hand, if $K\ge a^2$ for some positive constant a, then, just as in the Riemannian case, no geodesic segment of length greater than π/a can be locally minimizing.

3.2. Complete integrability. An important aspect of the theory of Finsler structures is the study of the geodesic flow, particularly methods for integrating the geodesic flow. I should emphasize that, just because one can write down the vector field $\mathbf{W}$ in some canonical coordinates, it does not follow that one can write down explicit first integrals for its flow, i.e., non-constant functions f which satisfy $df(\mathbf{W})=0$.

3.2.1. Symmetries as a source of first integrals. Classically, the source for first integrals of the geodesic flow is the algebra of *infinitesimal symmetries* of the contact form ω, i.e., vector fields $\mathbf{X}$ that satisfy

$$\mathfrak{L}_{\mathbf{X}}\,\omega=0.$$

These vector fields form a Lie algebra and the (local) flow of such a vector field fixes ω. Given such an infinitesimal symmetry $\mathbf{X}$, the function $f=\omega(\mathbf{X})$ satisfies

$$df=d\bigl(\omega(\mathbf{X})\bigr)=\mathfrak{L}_{\mathbf{X}}\,\omega-\mathbf{X}\lrcorner\bigl(\eta\wedge\theta\bigr)=-\eta(\mathbf{X})\,\theta+\theta(\mathbf{X})\,\eta,$$

[7] Every generalized Finsler structure is locally geodesically amenable.

so that $df(\mathbf{W}) = 0$.

Note that, unless $\mathbf{X}$ is a constant multiple of $\mathbf{W}$, df cannot vanish identically. For, suppose that $df \equiv 0$. Then $\eta(\mathbf{X}) = \theta(\mathbf{X}) = 0$, so that $\mathbf{X} = f\mathbf{W}$. Thus, every non-trivial infinitesimal symmetry other than $\mathbf{W}$ itself corresponds to a non-trivial first integral.

Conversely, suppose that f is a first integral of the geodesic flow, i.e., that $df(\mathbf{W}) = 0$. Then there are functions f_2 and f_3 so that $df = f_2\,\theta + f_3\,\eta$. By the structure equations, the vector field

$$\mathbf{X} = f\,\mathbf{W} + f_3\,\mathbf{T} - f_2\,\mathbf{H}$$

satisfies

$$\mathcal{L}_{\mathbf{X}}\,\omega = d\big(\mathbf{X} \lrcorner\, \omega\big) + \mathbf{X} \lrcorner\, \big(\eta\wedge\theta\big) = df - df = 0$$

and $\omega(\mathbf{X}) = f$. Thus, $\mathbf{X}$ is an infinitesimal symmetry of the contact form ω. However, it will not be true, in general, that $\mathbf{X}$ is an infinitesimal symmetry of the full coframing defining the generalized Finsler structure.

3.2.2. A second first integral. If two independent first integrals of the geodesic flow can be found, the flow is said to be *completely integrable*. For, say, if f_1 and f_2 are independent first integrals, then each simultaneous level set $f_i = c_i$ will be a union of integral curves of $\mathbf{W}$ and one can say that the integral curves of $\mathbf{W}$ are known implicitly.

In fact, however, a slightly weaker notion turns out to work almost as well: Because of the contact structure in the problem, having one first integral is almost as good as having two.

Suppose that f is a first integral of the geodesic flow and let U be the open set on which df is non-zero. Then, on U, one can write

$$d\omega = \eta\wedge\theta = df\wedge\phi$$

for some 1-form ϕ which is well-defined on U up to a multiple of df. Computing the exterior derivative of both sides of this equation yields $0 = -df\wedge d\phi$, so that $d\phi \equiv 0 \mod df$. In particular, on each level set $f = c$ where c is a regular value of f, the 1-form ϕ is closed. Thus, on such a level set, one can write $\phi = dg_c$ where g_c is a function whose level curves on $f = c$ are integral curves of $\mathbf{W}$. Note that g_c can be found by quadrature. Thus, up to quadrature, knowing one non-trivial first integral of the geodesic flow determines a second.

For this reason, when at least one non-trivial first integral of the geodesic flow can be found, the geodesic flow is said to be completely integrable.

3.2.2. Geometric conditions producing first integrals. Several natural classes of generalized Finsler structures have completely integrable geodesic flows.

For example, by the Bianchi identity ($2a$), the Landsberg structures, defined by the condition $C = 0$, all have the invariant S as a first integral of the geodesic flow. Thus, except for the subclass for which S is constant, the geodesic flow of a Landsberg structure is completely integrable.

Another such class is the class of structures for which K is constant. In that case, $K_1 = K_3 = 0$, so the Bianchi identities imply that the function $J = KS^2 + C^2$

is constant along the integral curves of $\mathbf{W}$. The tableau of free derivatives of this class of structures has characters $s_1 = 2$, $s_2 = 2$, and $s_3 = 0$; is involutive; and has vanishing torsion cokernel. Thus, up to diffeomorphism, these structures depend on two arbitrary functions of one variable. For the generic such structure, the function J will be non-constant, and the geodesic flow of such a structure will be completely integrable. This fact will play an important role in the analysis in the later sections of this manuscript.

These classes of Finsler structures provide new examples of what É. Cartan called equations of 'classe C' [Ca3]. Roughly speaking, a class of ODE which is invariant under some group of diffeomorphisms is said by Cartan to be of 'classe C' if first integrals for a member of the given class of ODE can be constructed from the differential invariants of the class under the group of diffeomorphisms. Thus, the geodesic flow arising from Landsberg Finsler structures or Finsler structures with constant K are examples of 'classe C'.

4. Structures satisfying $K = 0$

4.1. A local normal form. As was seen in §2, the generalized Finsler structures satisfying $K = 0$ depend on two arbitrary functions of two variables up to diffeomorphism. I will now derive a local normal form which exhibits this dependence.

Proposition. *Let (ω, θ, η) be a generalized Finsler structure on a 3-manifold Σ which satisfies $K \equiv 0$. Then every $\mathbf{u}$ in Σ has a neighborhood U on which there exist local coordinates (x, y, z) in which the coframing takes the form*

$$\begin{aligned} \omega &= dy - x\,dz \\ \theta &= F^{-1}\,dx + (Fy + H)\,dz \\ \eta &= F\,dz \end{aligned}$$

where $F \neq 0$ and H are functions of the variables x and z.

Conversely, if $F \neq 0$ and H are arbitrary functions of the variables x and z on some domain D in xz-space, then the above formulae define a generalized Finsler structure on $D \times \mathbb{R}$ in xzy-space which satisfies $K \equiv 0$ and

$$\begin{aligned} C &= F_x, \\ S &= -F^{-2} F_z + H_x + y\,F_x. \end{aligned}$$

Proof. First, note that under the assumption that $K \equiv 0$, the structure equations become

$$\begin{aligned} d\omega &= \quad \eta \wedge \theta \\ d\theta &= -\eta \wedge \big(\omega + S\,\theta\big) \\ d\eta &= -C\,\eta \wedge \theta \end{aligned}$$

In particular, $\eta \wedge d\eta \equiv 0$, so there exist functions $F \neq 0$ and z in a neighborhood of $\mathbf{u}$ so that $\eta = F\,dz$. Moreover, since $\eta \wedge \theta = d\omega$ is closed, there also exists a function x on a (possibly) smaller neighborhood so that $\eta\wedge\theta = dz\wedge dx$. It follows that there must exist a function G on this neighborhood so that

$$\theta = F^{-1}\,dx + G\,dz.$$

Also, since $d\omega = dz\wedge dx$, it follows that there exists a function y on a (possibly) still smaller neighborhood so that $\omega = dy - x\,dz$. Since $dy\wedge dz\wedge dx = \omega\wedge d\omega \neq 0$, it follows that the three functions x, y, and z are independent on this neighborhood. By shrinking the neighborhood one more time, I can suppose that $(x,y,z)\bigl(U\bigr)$ is a coordinate cube in xyz-space.

Now, since $d\eta = dF\wedge dz = -C\,dz\wedge dx$, it follows that $F_y = 0$ and that $F_x = C$. In particular, F is a function of x and z alone. Expanding the second structure equation and comparing terms yields

$$(dG - F\,dy)\wedge dz + (S - F^{-2}F_z)\,dz\wedge dx = 0.$$

It follows from this that $G_y = F$ and, since $F_y = 0$, this implies that when I write $G = Fy + H$, then $H_y = 0$, so that H is a function of x and z alone. Substituting this formula for G into the above displayed equation yields

$$S = F^{-2}F_z + H_x + y\,F_x\,,$$

as desired.

Finally, I leave it to the reader to check that when $F \neq 0$ and H are chosen arbitrarily as functions of x and z, then the given formulae of the Proposition actually do satisfy the structure equations of a generalized Finsler structure satisfying $K = 0$ and, moreover, that the formulae for C and S in terms of the derivatives of these two arbitrary functions are valid. □

Using this normal form, it is easy to see how to construct many geodesically complete examples of generalized Finsler structures. For example, simply take F and H to be globally defined on the xz-plane and to satisfy the conditions that F, H, and F^{-1} are bounded on the entire plane. Then the corresponding generalized Finsler structure on xyz-space will be complete in every desired sense.

Furthermore, under these boundedness assumptions, the resulting generalized Finsler structure will be amenable, i.e., the $\omega\theta$-leaf space will have the structure of a smooth surface. The reason for this is that each $\omega\theta$ integral curve is of the form $(x,y,z) = \bigl(u'(t), u(t), t\bigr)$ where u is a solution of the second order equation

$$u'' + F(u',t)\bigl(F(u',t)u + H(u',t)\bigr) = 0.$$

Since F and H are bounded, this equation has an entire solution for any initial conditions. Thus, every $\omega\theta$ integral curve transversely intersects the plane $z = 0$ in a unique point, making this plane a section of the $\omega\theta$ foliation.

However, without further hypotheses on the functions F and H, the corresponding immersion $\nu : \Sigma \to TM$ will not have closed circles as images of the π-fibers. Certainly some hypothesis will suffice. For example, $F \equiv 1$ and $H \equiv 0$ give rise to an image $\nu(\Sigma) \subset T\mathbb{R}^2$ which is just the unit sphere bundle of a flat metric on $\mathbb{R}^2$.

4.2. A global classification. Consider the problem of classifying the generalized Finsler structures for which $K = 0$ on a compact connected 3-manifold Σ. As has already been remarked above, for such a structure the function S is linear on the integral curves of the geodesic flow. However, since Σ is compact, S is bounded on any such curve. This implies that S is constant on such curves and hence that C vanishes identically. The structure equations are thus

$$\begin{aligned} d\omega &= \quad \eta \wedge \theta \\ d\theta &= -\eta \wedge \big(\omega + S\,\theta\big) \\ d\eta &= 0 \\ dS &= S_2\,\theta + S_3\,\eta \end{aligned}$$

The general methods of Cartan predict that coframings satisfying these equations depend on one function of two variables up to diffeomorphism. I am now going to make this explicit in this case.

Let **W**, **T**, and **H** denote as usual the dual vector fields to the coframing $\big(\omega, \theta, \eta\big)$ and let $p : \tilde{\Sigma} \to \Sigma$ be the universal cover of Σ. Since $d\eta = 0$ and η is nowhere vanishing, it follows that there exists a smooth function $z : \tilde{\Sigma} \to \mathbb{R}$ so that $dz = p^*\eta$. Since Σ is compact, it follows that the flow on $\tilde{\Sigma}$ of the lift of a vector field on Σ is complete, and this fact applied to the lift of **H** shows that $z : \tilde{\Sigma} \to \mathbb{R}$ is a surjective submersion. In fact, using the flow of the lift of **H** to identify z-fibers, it follows that $z : \tilde{\Sigma} \to \mathbb{R}$ is actually a fiber bundle and hence $\tilde{\Sigma} = N \times \mathbb{R}$ for some simply connected surface N. Moreover, by the structure equations, $d\big(p^*\theta\big)$ and $d\big(p^*\omega\big)$ are in the ideal generated by $p^*\eta = dz$, so these 1-forms pull back to any z-fiber $z^{-1}(c) \subset \tilde{\Sigma}$ to be closed and dual to the (complete) lifts of the vector fields **W** and **T**. It follows that N is diffeomorphic to $\mathbb{R}^2$, so that $\tilde{\Sigma}$ is diffeomorphic to $\mathbb{R}^3$.

To complete a construction of coordinates on $\tilde{\Sigma}$, I proceed as follows: Let $\sigma : \mathbb{R} \to \tilde{\Sigma}$ be a section of z, i.e., $z\big(\sigma(t)\big) = t$. Then for any point $u \in \tilde{\Sigma}$, there is a path in $z^{-1}\big(z(u)\big) \simeq \mathbb{R}^2$ joining u to $\sigma\big(z(u)\big)$ and any two such are homotopic. Set

$$x(u) = \int_{\sigma\left(z(u)\right)}^{u} \theta$$

where the integral it taken along a path lying in $z^{-1}\big(z(u)\big)$. Since any two such are homotopic and since the pullback of $p^*\theta$ to $z^{-1}\big(z(u)\big)$ is closed, this integral is independent of the choice of path and defines a smooth function $x : \tilde{\Sigma} \to \mathbb{R}$. By construction, $dx \equiv \theta \mod dz$, so it follows that $p^*\big(\theta\wedge\eta\big) = dx\wedge dz$.

In particular,

$$d\big(p^*\omega + x\,dz\big) = p^*(d\omega) + dx \wedge dz = p^*(\eta \wedge \theta) + dx \wedge dz = -dx \wedge dz + dx \wedge dz = 0$$

so that there exists a function $y : \tilde{\Sigma} \to \mathbb{R}$ satisfying $p^*\omega = dy - x\,dz$. From the completeness of the lifted dual vector fields it follows that the map $(x, y, z) : \tilde{\Sigma} \to \mathbb{R}^3$ is a diffeomorphism and so I will now regard these functions as coordinates on $\tilde{\Sigma}$.

Now $p^*\theta = dx + f\,dz$ for some function f on $\tilde\Sigma$ and the structure equations give

$$df \wedge dz = d\big(p^*\theta\big) = -p^*\eta \wedge \big(p^*\omega + p^*(S\,\theta)\big) = -dz \wedge \big(dy + p^*S\,dx\big),$$

so $d(f-y)\wedge dz = p^*S\,dx\wedge dz$. It follows that there exists a function h on $\mathbb{R}^2$ so that $f - y = h(x,z)$. Thus, the following formulae hold

$$\begin{aligned} p^*\omega &= dy - x\,dz \\ p^*\theta &= dx + \big(y + h(x,z)\big)\,dz \\ p^*\eta &= dz \\ p^*S &= h_x(x,z) \end{aligned}$$

Since some arbitrary choices were made in the definition of the coordinates (x,y,z) they are not unique and it is useful to examine how many ways coordinates satisfying these conditions can be introduced. Thus, suppose that

$$\begin{aligned} dy - x\,dz &= dy' - x'\,dz' \\ dx + \big(y + h\big)\,dz &= dx' + \big(y' + h'\big)\,dz' \\ dz &= dz' \end{aligned}$$

Then clearly there exists a constant c so that $z' = z + c$ and the relation $dy\wedge dz = dy'\wedge dz' = dy'\wedge dz$ implies that there exists a function ϕ of one variable so that $y' = y + \phi(z)$. Then the second relation implies that $x' = x + \phi'(z)$.

Thus, coordinates of the above kind are unique up to diffeomorphisms Φ of the form

$$\Phi\big(x,y,z\big) = \big(x{+}\phi'(z),\ y{+}\phi(z),\ z{+}c\big).$$

These diffeomorphisms form a pseudo-group Γ on $\mathbb{R}^3$ whose general element depends on one function of one variable and one constant. By changing variables via this pseudo-group, one cannot much further normalize the function h of two variables.

Conversely, starting with any function h on $\mathbb{R}^2$, the equations

$$\begin{aligned} \bar\omega &= dy - x\,dz \\ \bar\theta &= dx + \big(y + h(x,z)\big)\,dz \\ \bar\eta &= dz \\ \bar S &= h_x(x,z) \end{aligned}$$

satisfy the structure equations

$$\begin{aligned} d\bar\omega &= \quad \bar\eta \wedge \bar\theta \\ d\bar\theta &= -\bar\eta \wedge \big(\bar\omega + \bar S\,\bar\theta\big) \\ d\bar\eta &= 0, \end{aligned}$$

which verifies the earlier claim that the coframings of this type depend on one function of two variables up to diffeomorphism.

It remains to be seen what restrictions are placed on h by the requirement that there exist a group Λ acting freely and discretely on $\mathbb{R}^3$ which preserves the coframing $\bigl(\bar\theta,\bar\omega,\bar\eta\bigr)$ and has the property that $\mathbb{R}^3/\Lambda$ is compact.

By the discussion above, any element of the group of symmetries of $\bigl(\bar\theta,\bar\omega,\bar\eta\bigr)$ is an element of Γ. I will write such elements in the form $\langle\phi,c\rangle$ where ϕ is a function of one variable and c is a constant. The action of Γ on $\mathbb{R}^3$ will be written

$$\langle\phi,c\rangle\cdot\bigl(x,y,z\bigr)=\bigl(x{+}\phi'(z),\ y{+}\phi(z),\ z{+}c\bigr).$$

The product in this group is given by $\langle\phi,a\rangle\langle\psi,b\rangle=\langle\phi+\psi_a,\ a{+}b\rangle$ where I define ϕ_a to be the function satisfying $\phi_a(z)=\phi(z+a)$.

Now, if $\langle\phi,c\rangle$ it is to preserve the given coframing, it must satisfy

$$h\bigl(x{+}\phi'(z),\ z+c\bigr)+\phi(z)+\phi''(z)=h(x,z)$$

In other words, the graph $w=h(x,z)$ is invariant under the Λ-action

$$\langle\phi,c\rangle\cdot(x,z,w)=\bigl(x{+}\phi'(z),\ z{+}c,\ w{-}\phi''(z){-}\phi(z)\bigr).$$

First, if all of the elements in Λ were of the form $\langle\phi,0\rangle$, then $\mathbb{R}^3/\Lambda$ could not be compact. Thus, I may choose an element $\langle\phi,c\rangle\in\Lambda$ with $c>0$. Since

$$\langle\psi,0\rangle^{-1}\langle\phi,c\rangle\,\langle\psi,0\rangle=\langle\phi+\psi_c-\psi,c\rangle,$$

it follows that conjugating the choice of coordinates by an element of the form $\langle\psi,0\rangle$ where $\phi=-\psi_c+\psi$ will replace h by a new h for which $h(x,z+c)=h(x,z)$, i.e., h becomes c-periodic in its second variable. I will assume from now on that this has been done.

There are then two cases, either h does not depend on z at all or else there is a minimum $p>0$ so that $h(x,z+p)=h(x,z)$ and c must be an integer multiple of p.

In the former case, the reader will find that the requirement there be a co-compact subgroup of Γ which satisfies

$$h\bigl(x+\phi'(z)\bigr)+\phi''(z)+\phi(z)=h(x)$$

forces h to be linear, so that $h=Sx+h_0$ and a further conjugation reduces to the case $h=Sx$. Thus, in this case, the group of symmetries of the coframing on $\mathbb{R}^3$ is simply transitive. Indeed, $\mathbb{R}^3$ can be given the structure of a Lie group G_S in a unique way so that $(0,0,0)$ is the identity element and the given coframing consists of left-invariant 1-forms. Then the question reduces to classifying the co-compact lattices in G_S, a well-understood problem.

It remains to treat the case where h actually depends on z. Crucial for this is understanding the equation for $h_x=S$. I will return to this later. What I want to show is that the existence of a co-compact lattice forces S to be independent of x, which will ultimately make these be like Minkowski models.

5. Structures satisfying $K = 1$

In this section, I want to study the (generalized) Finsler structures that satisfy $K = 1$ and construct examples of such on the standard 2-sphere.

Thus, let (ω, θ, η) be a generalized Finsler structure on a connected 3-manifold Σ that satisfies $K \equiv 1$. In this case, the structure equations become

$$\begin{aligned} d\omega &= \quad \eta \wedge \theta \\ d\theta &= -\eta \wedge (\omega + S\,\theta) \\ d\eta &= -(\omega + C\,\eta) \wedge \theta \end{aligned}$$

$$\begin{aligned} dS &= \quad C\,\omega + S_2\,\theta + S_3\,\eta \\ dC &= -S\,\omega + C_2\,\theta + C_3\,\eta \end{aligned}$$

Let $\mathbf{W}$, $\mathbf{T}$, and $\mathbf{H}$ denote as usual the dual vector fields to the coframing (ω, θ, η). Note that $J = S^2 + C^2$ is constant along the geodesic curves, i.e., the integral curves of $\mathbf{W}$, and so is a first integral of the geodesic equations. More interesting than this function, however, are the 1-form $S\,\theta + C\,\eta$, the 2-form $\eta\wedge\theta$, and the quadratic form $\eta^2 + \theta^2$. A computation from the structure equations yields

$$\mathfrak{L}_{\mathbf{W}}(S\,\theta + C\,\eta) = \mathfrak{L}_{\mathbf{W}}(\eta \wedge \theta) = \mathfrak{L}_{\mathbf{W}}(\eta^2 + \theta^2) = 0,$$

so that all of these quantities are invariant under the geodesic flow.

In fact, $\mathfrak{L}_{\mathbf{W}}\,\omega = 0$, $\mathfrak{L}_{\mathbf{W}}\,\theta = \eta$, and $\mathfrak{L}_{\mathbf{W}}\,\eta = -\theta$, which implies

$$\begin{aligned} \exp^*_{t\mathbf{W}}\,\omega &= \omega \\ \exp^*_{t\mathbf{W}}\,\theta &= \quad \cos t\,\theta + \sin t\,\eta \\ \exp^*_{t\mathbf{W}}\,\eta &= -\sin t\,\theta + \cos t\,\eta \end{aligned}$$

These formulae have some interesting consequences for geodesically complete structures with $K \equiv 1$. For example, set $\Phi = \exp_{2\pi\mathbf{W}}$. Then $\Phi : \Sigma \to \Sigma$ is a symmetry of the generalized Finsler structure. If any geodesic closes with length an integral multiple of 2π, then this implies that some power of Φ, say Φ^k has a fixed point. Since Φ^k is a symmetry of the coframing and Σ is connected, this implies that Φ^k is the identity map. Thus, having one closed geodesic of length an integer multiple of 2π implies that all of the geodesics are closed and have the same length.[8]

Another interesting observation is that, if Σ is compact, then so is the group G of symmetries the generalized Finsler structure. It follows that either Φ^k is the identity for some $k > 0$, so that all of the geodesics close, or else the closure of the set $\{\Phi^k \mid k \geq 0\}$ in G contains an abelian sub-group of positive dimension. Now, it can be shown that there cannot be a compact example with a 2-dimensional abelian subgroup of the group of symmetries, so the only possibilities for a compact example with positive dimensional symmetry group are the quotients of the standard Riemannian structure on the 3-sphere and examples with a 1-parameter symmetry group. I will construct examples of this latter type at the end of this section.

[8]Actually, if any geodesic on which J is non-zero closes, then it must have length an integral multiple of 2π. This follows from the formulae $\mathfrak{L}_{\mathbf{W}}\,S = C$ and $\mathfrak{L}_{\mathbf{W}}\,C = -S$, which imply that S and C are periodic of period 2π with respect to arc length along any geodesic.

5.1. Canonical structures on the geodesic space. Suppose that $U \subset \Sigma$ is an open set in which the geodesic foliation is amenable, i.e., the leaf space Λ of the integral curves of $\mathbf{W}$ in U carries the structure of a Hausdorff smooth manifold so that the quotient map $\ell : U \to \Lambda$ is a smooth submersion. Then there exist a 1-form φ on Λ so that $\ell^*\varphi = -S\,\theta - C\,\eta$; a 2-form dA on Λ so that $\ell^*(dA) = \eta\wedge\theta$; and a positive definite quadratic form g on Λ so that $\ell^* g = \eta^2 + \theta^2$. In particular, when the space of geodesics is a manifold, it has a canonical metric g and orientation dA as well as a canonical 1-form φ, whose norm with respect to the metric g is the invariant J.

5.2. Recovering the generalized Finsler structure. In fact, starting with this data on a surface Λ, one can recover the generalized Finsler structure. To see this, let $\nu : F \to \Lambda$ denote the oriented orthonormal frame bundle of Λ with respect to the metric g and orientation form dA. One can then define a map $\hat\ell : U \to F$ by the formula

$$\hat\ell(u) = \big(\ell(u);\, \ell_*(\mathbf{H}_u),\, \ell_*(\mathbf{T}_u)\big) \in F_u\,.$$

(The facts that $(\eta\wedge\theta)(\mathbf{H},\mathbf{T}) \equiv 1$ and that $\mathbf{H}$ and $\mathbf{T}$ are the duals to η and θ ensures that $\big(\ell_*(\mathbf{H}_u),\, \ell_*(\mathbf{T}_u)\big)$ is a dA-oriented, g-orthonormal frame at $\ell(u) \in \Lambda$.) By the very definition of the canonical 1-forms α_1 and α_2 on F, it follows that $\hat\ell^*(\alpha_1) = \eta$ and $\hat\ell^*(\alpha_2) = \theta$. The defining equations of the Levi-Civita connection 1-form α_{21} on F are

$$d\alpha_1 = \alpha_{21}\wedge\alpha_2 \qquad\qquad d\alpha_2 = -\alpha_{21}\wedge\alpha_1$$

so pulling these back via $\hat\ell$ yields

$$d\eta = \hat\ell^*(\alpha_{21})\wedge\theta \qquad\qquad d\theta = -\hat\ell^*(\alpha_{21})\wedge\eta.$$

Comparing this with the structure equations above then yields

$$\hat\ell^*(\alpha_{21}) = -\big(\omega + S\theta + C\eta\big) = -\omega + \ell^*\varphi.$$

In particular, $\hat\ell^*\big(\alpha_1\wedge\alpha_2\wedge\alpha_{21}\big) = -\eta\wedge\theta\wedge\omega \neq 0$, so $\hat\ell$ is a local diffeomorphism. If I denote the Gauss curvature of the metric g by R, then the usual structure equation yields $d\alpha_{21} = \nu^*\big(-R\,dA\big)$. In particular, computing the exterior derivative of the equation $\hat\ell^*(-\alpha_{21} + \nu^*\varphi) = \omega$ yields

$$\ell^*(dA) = \eta\wedge\theta = d\omega = \hat\ell^*\big(-d\alpha_{21} + \nu^*(d\varphi)\big) = \hat\ell^*\nu^*\big(R\,dA + d\varphi\big) = \ell^*\big(R\,dA + d\varphi\big),$$

so that $d\varphi = (1{-}R)\,dA$.

Conversely, suppose now that I start with an oriented surface Λ endowed with a Riemannian metric g with Gauss curvature R and a 1-form φ which satisfies $d\varphi = (1{-}R)\,dA$. Then on the orthonormal frame bundle $\nu : F \to \Lambda$ with tautological 1-forms α_1 and α_2 and Levi-Civita connection 1-form α_{21} satisfying the standard structure equations

$$\begin{aligned} d\alpha_1 &= \alpha_{21}\wedge\alpha_2 \\ d\alpha_2 &= -\alpha_{21}\wedge\alpha_1 \\ d\alpha_{21} &= -\nu^* R\,\alpha_1\wedge\alpha_2 \end{aligned}$$

write $\nu^*\varphi = -C\,\alpha_1 - S\,\alpha_2$ for some functions S and C on F and consider the 1-forms

$$\begin{aligned}\bar\omega &= -\alpha_{21} + \nu^*\varphi = -\alpha_{21} - C\,\alpha_1 - S\,\alpha_2\\ \bar\theta &= \quad \alpha_2\\ \bar\eta &= \quad \alpha_1\end{aligned}$$

Then these 1-forms and functions on F satisfy the structure equations

$$\begin{aligned}d\omega &= \quad \eta\wedge\theta\\ d\theta &= -\eta\wedge\bigl(\omega + S\,\theta\bigr)\\ d\eta &= -\bigl(\omega + C\,\eta\bigr)\wedge\theta\end{aligned}$$

and so correspond to a generalized Finsler structure with $K = 1$. Thus, the given data (g, dA, φ) on the surface Λ suffice to determine a local solution to the $K = 1$ equation.

5.2.1. A normal form. This local picture is still not entirely satisfactory because of the need to solve the equation $d\varphi = (1-R)\,dA$. However, this last step can also be avoided in a more-or-less natural way. The metric g is conformal to a metric $\bar g$ of constant Gauss curvature $\bar R$, say $g = e^{2u}\,\bar g$. There is a natural map from the oriented orthonormal frame bundle $\bar F$ of $\bar g$ to F that simply scales a $\bar g$-orthonormal frame by e^{-u}. I will denote this map by $\mu : \bar F \to F$. Then the standard formulae yield

$$\begin{aligned}\mu^*\alpha_1 &= e^u\,\bar\alpha_1\\ \mu^*\alpha_2 &= e^u\,\bar\alpha_2\\ \mu^*\alpha_{21} &= \bar\alpha_{21} + {*du}\end{aligned}$$

Pulling back the equation $d\alpha_{21} = \nu^*\bigl(d\varphi - dA\bigr)$ via μ then yields

$$d\bigl(\bar\alpha_{21} + {*du}\bigr) = \bar\nu^*\bigl(d\varphi - e^{2u}d\bar A\bigr),$$

so setting $\bar\varphi = \varphi - {*du}$, this can be written

$$d\bar\varphi + \bar R\,d\bar A = e^{2u}d\bar A.$$

This yields the following prescription: Start with a metric $\bar g$ of constant curvature $\bar R$ and choose any 1-form $\bar\varphi$ satisfying the open condition $d\bar\varphi + \bar R\,d\bar A > 0$. Then define the function u by the equation $d\bar\varphi + \bar R\,d\bar A = e^{2u}d\bar A$. Then the 1-forms

$$\begin{aligned}\bar\omega &= -\bar\alpha_{21} + \nu^*\bar\varphi\\ \bar\theta &= e^u\,\bar\alpha_2\\ \bar\eta &= e^u\,\bar\alpha_1\end{aligned}$$

will satisfy the structure equations of a generalized Finsler structure with $K = 1$.

Explicitly, in the case $\bar{R} = 0$, let x and y be local coordinates on $V \subset \Lambda$ so that, on V, one has $\bar{g} = dx^2 + dy^2$ and $d\bar{A} = dx\wedge dy$. Choose a 1-form $\bar{\varphi} = a\,dx + b\,dy$ satisfying the open condition $(b_x - a_y) > 0$. Then on $V \times S^1 \simeq \bar{F}_V$, the 1-forms

$$\begin{aligned}
\bar{\omega} &= -d\phi + a\,dx + b\,dy \\
\bar{\theta} &= \sqrt{b_x - a_y}\,\bigl(-\sin\phi\,dx + \cos\phi\,dy\bigr) \\
\bar{\eta} &= \sqrt{b_x - a_y}\,\bigl(\quad\cos\phi\,dx + \sin\phi\,dy\bigr)
\end{aligned}$$

satisfy the structure equations of a generalized Finsler structure with $K = 1$ and every generalized Finsler structure with $K = 1$ is locally of this form. Thus, the general local solution is explicitly given in terms of two arbitrary functions a and b of two variables, as the general theory predicted.

5.3. Compact examples. I now want to make some remarks about compact examples and Finsler structures satisfying $K = 1$ on the 2-sphere.

5.3.1. The Riemannian case. The case where S (and hence, C) vanishes identically is the (generalized) Riemannian case, for which there is a unique local model. If Σ is compact, then the simply connected cover $\tilde{\Sigma}$ is diffeomorphic to the 3-sphere via a diffeomorphism which identifies the Finsler coframing to the standard left-invariant coframing of S^3 thought of as the Lie group SU(2). Thus $\Sigma = \Gamma\backslash\,\mathrm{SU}(2)$ where $\Gamma \subset \mathrm{SU}(2)$ is a finite subgroup.

5.3.2. The non-Riemannian case. For the rest of the discussion, I will suppose that S does not vanish identically.

Since $dS \equiv C\omega \mod \{\theta, \eta\}$ and $dC \equiv -S\omega \mod \{\theta, \eta\}$, it follows that S cannot be constant. Thus, the group of symmetries of the coframing cannot act transitively. In particular, the dimension of this group must be at most 2. Recall that in the Riemannian case, the group of symmetries of a Riemannian metric may never be of dimension 2.

The first problem is how to describe the actual Finsler structures with $K = 1$. As will be seen, this is a more delicate matter than just solving the local generalized Finsler structure problem as was done above.

To see what the conditions should be, first suppose that $\Sigma \subset TM$ is an actual Finsler structure satisfying $K = 1$ and let ω, θ, and η denote the canonical 1-forms satisfying the structure equations as above. Then the fibers of the projection $\pi : \Sigma \to M$ are smooth embedded circles and are the leaves of the system $\omega = \theta = 0$. Thus, an obvious necessary condition for a generalized Finsler structure to be a Finsler structure is that all the leaves of the system $\omega = \theta = 0$ be compact. This is not sufficient however because of two difficulties.

The first difficulty is that even though all of the leaves may be compact, so that the system $\omega = \theta = 0$ defines a foliation by circles, there may exist certain exceptional circles (necessarily isolated) around which the foliation is not locally product of a circle and a 2-disk. Such 'multiply covered' circles cause the leaf space M to have the structure of an orbifold. One can deal with this either by considering Finsler structures on orbifolds, which does not significantly change the local theory, or by simply deleting the offending circles, yielding a smooth quotient space.

The second difficulty is that even if the foliation is locally a product, with leaf space quotient $\pi : \Sigma \to M$, the map $\iota : \Sigma \to TM$ may immerse each fiber Σ_x into T_xM as a convex curve which winds around the origin more than once, say $m > 1$ times. If the images $\iota\big(\Sigma_x\big) \subset T_xM$ are all simple embedded curves, then one can divide Σ by an action of $\mathbb{Z}_m$ so as to get a new Σ' which is a Finsler structure on M. However, there is no reason a priori for the images to be simple closed curves, even in the case where $K = 1$, as will be seen.

5.3.3. The local non-Riemannian case. Returning to the case of a Finsler structure $\Sigma \subset TM$, let $D \subset M$ be a geodesically convex disk with smooth strictly convex boundary ∂D. (Such disks exist, for example, the geodesic ball $B_r(x)$ satisfies this condition for sufficiently small r.) Then an (oriented) geodesic γ through any point in the interior meets the boundary ∂D in two distinct points γ_- (entering) and γ_+ (exiting). Conversely, for any two distinct points p_- and p_+ in ∂D, there is a unique geodesic segment γ lying in D and starting at p_- and ending at p_+. Thus, the space Λ_D of geodesic segments in D is topologically (in fact, smoothly) identifiable with $\big(\partial D \times \partial D\big) \setminus \Delta$, a smooth surface diffeomorphic to a cylinder. Let $\Sigma_D = \pi^{-1}(D)$. By the remarks just made, the geodesic foliation on Σ_D is amenable, with a smooth submersion $\ell : \Sigma_D \to \Lambda_D$ having the leaves of this foliation as fibers. This yields the double fibration

$$\begin{array}{ccccc} & & \Sigma_D & & \\ & {\scriptstyle \ell}\swarrow & & \searrow{\scriptstyle \pi} & \\ \Lambda_D & & & & D. \end{array}$$

Note that for any $x \in D$, the set $C_x = \ell\big(\pi^{-1}(x)\big) \subset \Lambda_D$ consists of the oriented geodesics which pass through x, and the map $\ell : \pi^{-1}(x) \to \Lambda_D$ is a smooth embedding of a circle into Λ_D which generates the fundamental group of $\Lambda_D \simeq S^1 \times (0, 1)$.

Now, assuming $K \equiv 1$ for the original Finsler structure, let g, dA and φ be the canonical metric, area form, and 1-form, respectively, constructed earlier on Λ_D. Identifying Σ_D with an open subset of F, the oriented orthonormal frame bundle of L_D via the map $\hat{\ell}$ constructed earlier, the curves C_x are the images of the (closed) integral curves of the system $\alpha_2 = \alpha_{21} - \nu^*\varphi = 0$.

The geometric interpretation of these integral curves is as follows: For any unit speed (oriented) curve $\gamma : [a, b] \to \Lambda_D$, there are two natural functions associated with it. First, there is the geodesic curvature $\kappa : [a, b] \to \mathbb{R}$ and, second, there is the function $f : [a, b] \to \mathbb{R}$ defined by $f(t) = \varphi\big(\gamma'(t)\big)$. I will say that γ is a φ-geodesic if $\kappa = f$. The notion of 'λ-geodesic' would make sense for any 1-form λ, and the 0-geodesics are, of course, the usual geodesics.[9] Note that the λ-geodesics are solutions of a second order ODE and that there is exactly one oriented, unit-speed λ-geodesic in each tangent direction through each point of Λ_D.

By this definition, the curves C_x are φ-geodesics. Moreover, each closed φ-geodesic C is the projection of a unique closed integral curve of the system $\alpha_2 = \alpha_{21} - \nu^*\varphi = 0$ which must therefore constitute a single π-fiber, say $\pi^{-1}(x)$. Of course, this implies that $C = C_x$ and hence that C is embedded and represents a

[9] I am told that the 'physical interpretation' of λ-geodesics is that of a particle moving under the influence of a 'magnetic field' specified by λ.

generator of the fundamental group of Λ_D. Note also that for each Finsler geodesic γ in D, the interval of points y in D which lie on γ give rise to an interval of closed φ-geodesics C_y which pass through γ when it is regarded as a point of Λ_D. Thus, each point of Λ_D has an open interval of tangent directions for which the corresponding φ-geodesic is closed and represents a generator of the fundamental group of Λ_D.

This leads to the following prescription for constructing local Finsler structures with $K = 1$. Since Λ_D is an annulus, the Uniformization Theorem implies that $\bigl(\Lambda_D, g\bigr)$ is conformally equivalent to a standard annulus. In other words, there are coordinates r and ψ (well-defined modulo 2π) on Λ_D so that $g = e^{2u}\,(dr^2 + d\psi^2)$ and $dA = e^{2u}\,dr{\wedge}d\psi$ and so that the image $(r,\psi)\bigl(\Lambda_D\bigr)$ in $\mathbb{R}\times S^1$ is a standard cylinder of the form $I\times S^1$ where $I\subset\mathbb{R}$ is an open interval. By the previous local discussion, I can then embed Σ_D as an open subset of $I\times S^1\times S^1$ (with coordinates (r,ψ,ϕ)) in such a way that, for some functions a and b of r and ψ with $(b_r{-}a_\psi)>0$ the following formulae hold,

$$
\begin{aligned}
\omega &= \,-\,d\phi + a\,dr + b\,d\psi\\
\theta &= \sqrt{b_r{-}a_\psi}\,\bigl(-\sin\phi\,dr + \cos\phi\,d\psi\bigr)\\
\eta &= \sqrt{b_r{-}a_\psi}\,\bigl(\quad\cos\phi\,dr + \sin\phi\,d\psi\bigr)
\end{aligned}
$$

The functions a and b must then be chosen so that an open subset of the integral curves of the system $\theta=\omega=0$ are closed and project to $I{\times}S^1$ to become generators of the fundamental group.

5.3.4. Rotationally invariant examples. To show that this is possible for something other than the Riemannian case, I am now going to construct some examples, adapting an idea found in Volume 3 of Darboux [Da] for constructing metrics on S^2, all of whose geodesics are closed.[10] The simplest way to proceed to these examples is to try to find ones which are rotationally symmetric with respect to the cyclic coordinate ψ, i.e., so that $a_\psi = b_\psi = 0$. In fact, I am going to make the even more stringent assumption that $a = 0$, so that the forms involved simplify to

$$
\begin{aligned}
\omega &= \,-\,d\phi + b(r)\,d\psi\\
\theta &= \sqrt{b'(r)}\,\bigl(-\sin\phi\,dr + \cos\phi\,d\psi\bigr)\\
\eta &= \sqrt{b'(r)}\,\bigl(\quad\cos\phi\,dr + \sin\phi\,d\psi\bigr)
\end{aligned}
$$

where, by assumption $b'(r) > 0$ for all r in the interval I, so I will write $b = B'(r)$ where B is a strictly convex function on I. A computation shows that the invariant S is given by

$$
S = -\frac{b''(r) + 2b(r)b'(r)}{2\bigl(b'(r)\bigr)^{3/2}}\,\cos\phi.
$$

[10]Such metrics are called Zoll metrics. The interested reader might want to consult Besse's treatise [Be] for further developments of these ideas, both for surfaces and for higher dimensional manifolds.

Thus, as long as $b' + b^2$ is not constant, the examples constructed will not be Riemannian.

Now, the system $\omega = \theta = 0$ has a first integral. To see this, note that

$$b(r)\,\bigl(-\sin\phi\,dr + \cos\phi\,d\psi\bigr) - \cos\phi\,\bigl(-d\phi + b(r)\,d\psi\bigr) = d\bigl(\sin\phi\bigr) - \sin\phi\,\bigl(b(r)\,dr\bigr)$$
$$= e^{B}\,d\bigl(e^{-B}\sin\phi\bigr).$$

Thus every integral curve of this system lies on a level surface of the function $f = e^{-B}\sin\phi$. In particular, the zero level surface is foliated by the curves defined by $\psi \equiv \psi_0$ and $\phi \equiv 0$ or π. None of these curves are closed, of course, so I must look at the other level sets. Since the system is invariant under the involution $(r, \psi, \phi) \mapsto (r, -\psi, -\phi)$, it suffices to look at the positive level surfaces of f.

First, suppose that b does not vanish on the interval I. Then B has a smooth inverse on I. Let C be an integral curve of θ and ω on which f is positive, say $f \equiv e^{-B_0}$. The equation $e^{-B}\sin\phi = e^{-B_0}$ can be solved on C for r in the form

$$r = B^{-1}\bigl(e^{B_0}\sin\phi\bigr).$$

Since $\sin\phi$ must be positive on C, it follows that $0 < \phi < \pi$ on C. Moreover, I claim that ϕ must be monotone on C, which implies that it cannot be closed. To see this, note that because r is a function of ϕ on C and C is immersed, then at least one of $d\phi$ or $d\psi$ is non-vanishing at every point. However, since b is non-vanishing and $\omega = -d\phi + b\,d\psi$, it follows that neither $d\phi$ nor $d\psi$ can vanish. Thus, ϕ is monotone as claimed, and there are no closed integral curves.

Thus, from now on, I assume that b does vanish somewhere on I. Since $b_r > 0$, this zero must be unique. By translation in the variable r, I can assume that $b(0) = 0$ and can also make B unique by setting $B(0) = 0$. Since $B'' > 0$, it follows that there exists a unique smooth function ρ of r so that $\rho' > 0$, $\rho(0) = 0$, and $e^{-B} = \cos\rho$. Note that ρ satisfies $-\pi/2 < \rho < \pi/2$. I will let $\rho_{\min}$ denote the infimum of ρ and $\rho_{\max}$ denote the supremum of ρ. I am now going to use ρ as the natural parameter on I, so that I can regard r as a function of ρ, instead of the other way around and write $\dot r$ for $dr/d\rho$. Thus, the system to be integrated now is of the form

$$-\dot r\,\sin\phi\,d\rho + \cos\phi\,d\psi = -\dot r\,\cos\rho\,d\phi + \sin\rho\,d\psi = 0$$

In these coordinates, $f = \cos\rho\,\sin\phi \le 1$. The level set $f = 1$ is just $\rho = 0$ and $\phi = \pi/2$, which is a closed integral curve whose projection into $I \times S^1$ is a generator for the fundamental group. For any value ρ_0 so that $\rho_{\min} < -\rho_0 < 0 < \rho_0 < \rho_{\max}$, the equation $\cos\rho\,\sin\phi = \cos\rho_0 > 0$ is a closed curve C_{ρ_0} in $I \times S^1$. In particular, since $\cos\rho$ and $\sin\phi$ cannot vanish on C_{ρ_0}, it follows that on any integral curve of the system which lies in $f = \cos\rho_0$, the equation can be written in the form

$$d\rho = \frac{\cos\phi\,d\psi}{\dot r\,\sin\phi} \qquad\qquad d\phi = \frac{\sin\rho\,d\psi}{\dot r\,\cos\rho}\,,$$

so that $d\psi$ cannot vanish on any such integral curve, so I can regard it as a parameter on the curve. In fact, these equations show that the point $\bigl(\rho(\psi), \phi(\psi)\bigr)$ traces around the curve C_{ρ_0} counterclockwise as ψ increases. Thus, the integral curve will close if and only if ψ increases by an rational multiple of π when this point makes a complete circuit around C_{ρ_0}. A simple calculation reveals that the increase of ψ corresponding to such a complete circuit is

$$P(\rho_0) = \int_{-\rho_0}^{\rho_0} \frac{2\cos\rho_0\,\dot r(\rho)}{\sqrt{\cos^2\rho - \cos^2\rho_0}}\,d\rho\,.$$

The condition on r needed to satisfy the π_1-generation condition is that $P(\rho_0) \equiv 2\pi$ for all ρ_0 in the range $0 < \rho_0 < \pi/2$.

Now, it is known that for ρ_0 satisfying $0 < \rho_0 < \pi/2$

$$\int_{-\rho_0}^{\rho_0} \frac{2\cos\rho_0}{\cos\rho\,\sqrt{\cos^2\rho - \cos^2\rho_0}}\,d\rho \equiv 2\pi.$$

It follows that the necessary and sufficient condition that $P(\rho_0) \equiv 2\pi$ for all ρ_0 in the range $0 < \rho_0 < \pi/2$ is that $\dot r$ be of the form $\dot r = \bigl(1 + h(\rho)\bigr)/(\cos\rho)$ where h is an odd function defined on an interval symmetric about $\rho = 0$.

In order to have $\dot r > 0$ it will also be necessary to impose the condition $h > -1$. The identity $b\,dr = b\,\dot r\,d\rho = -e^B\,d\bigl(e^{-B}\bigr) = \cot\rho\,d\rho$ yields $b = \sin\rho/\bigl(1 + h(\rho)\bigr)$ and the condition $b_r = db/dr > 0$ becomes

$$1 + h(\rho) - h'(\rho)\,\tan\rho > 0,$$

which I also impose. Conversely, if h is an odd function on the interval $\bigl(-a, a\bigr)$ which satisfies these inequalities, the resulting generalized Finsler structure will have all of the $\omega\theta$-curves in the region $f > \cos a$ be closed.

5.3.5. Compact examples. Now, I further require that the odd function h be defined and smooth on all of $\mathbb{R}$, that it satisfy $h(\rho+\pi) = -h(\rho)$ and that h and h' vanish at $\rho = \pm\pi/2$. This will ensure that the formulae above define a smooth metric on S^2, all of whose φ-geodesics are closed. By the general procedure described above, as long as h is non-zero, this will induce a non-Riemannian Finsler structure satisfying $K \equiv 1$ on the 2-sphere of oriented φ-geodesics. (Writing this Finsler structure out explicitly would require finding a second first integral of the φ-geodesic flow, a non-trivial task.)

6. Structures satisfying $K = -1$

In this final section, I want to briefly consider the corresponding problem of studying the generalized Finsler structures which satisfy $K = -1$.

Thus, let $\bigl(\omega, \theta, \eta\bigr)$ be a generalized Finsler structure on a connected 3-manifold Σ that satisfies $K \equiv -1$. In this case, the structure equations become

$$\begin{aligned} d\omega &= \eta\wedge\theta\\ d\theta &= -\eta\wedge\bigl(\omega + S\,\theta\bigr)\\ d\eta &= \bigl(\omega - C\,\eta\bigr)\wedge\theta \end{aligned}$$

$$\begin{aligned} dS &= C\,\omega + S_2\,\theta + S_3\,\eta\\ dC &= S\,\omega + C_2\,\theta + C_3\,\eta \end{aligned}$$

These structure equations already have global implications. For example, suppose that Σ were compact. Then it would necessarily be geodesically complete, but the structure equations above show that, along any integral curve of $\mathbf{W}$, the function S satisfies an equation of the form $S'' - S = 0$, where the prime denoted differentiation with respect to the flow parameter along the integral curve. However, the only solution of this equation which is bounded in both directions is the zero solution. Since Σ is supposed to be compact, S must be bounded on Σ and hence on every integral curve. Of course this implies that S (and hence C) must vanish identically, so that the structure is locally Riemannian. In particular any Finsler structure on a compact surface M which satisfies $K \equiv -1$ (or any negative constant, for that matter) must be a Riemannian metric[11] , a result to be found in [Ak].

Note that $J = -S^2 + C^2$ is constant along the geodesic curves, i.e., the integral curves of $\mathbf{W}$, and so is a first integral of the geodesic equations. Thus, in the non-Riemannian case, the geodesic flow must be completely integrable, again, a great contrast with the Riemannian case when $K = -1$.

6.1. Canonical structures on the geodesic space. Consider the 1-form $S\,\theta - C\,\eta$, the 2-form $\eta{\wedge}\theta$, and the quadratic form $\eta^2 - \theta^2$. A computation from the structure equations yields

$$\mathfrak{L}_{\mathbf{W}}\big(S\,\theta - C\,\eta\big) = \mathfrak{L}_{\mathbf{W}}\big(\eta\wedge\theta\big) = \mathfrak{L}_{\mathbf{W}}\big(\eta^2 - \theta^2\big) = 0,$$

so that all of these quantities are invariant under the geodesic flow.

If one assumes that the generalized Finsler structure is geodesically amenable, with geodesic projection $\ell : \Sigma \to \Lambda$, then it follows that there exist on Λ a 1-form φ so that $\ell^*\varphi = -S\,\theta + C\,\eta$; a 2-form dA so that $\ell^*\big(dA\big) = \eta{\wedge}\theta$; and a Lorentzian quadratic form g so that $\ell^* g = \eta^2 - \theta^2$.

Using the same sort of immersion of Σ into the 'orthonormal frame bundle of g as I did in the $K = 1$ case, I can identify η and θ as the canonical forms on the Lorentzian orthonormal frame bundle of g and, due to the equations

$$\begin{aligned} d\eta &= (\omega + S\,\theta - C\,\eta)\wedge\theta \\ d\theta &= (\omega + S\,\theta - C\,\eta)\wedge\eta \end{aligned}$$

one sees that $\psi = (\omega{+}S\,\theta{-}C\,\eta)$ can be thought of as the Levi-Civita connection of this pseudo-metric. The curvature R of this metric is then defined by $d\psi = R\,\eta{\wedge}\theta$ and is well-defined on Λ. Then, just as before, one derives

$$d\varphi = (1 - R)\,dA.$$

as the equation relating the 1-form φ with the oriented Lorentzian structure defined by g and the choice of oriented area form dA.

Conversely, starting with an oriented surface Λ endowed with a Lorentzian metric g of curvature R and area form dA and a 1-form φ which satisfies the

[11]Note that this result definitely does not hold for Finsler structures on a compact surface if one merely assumes that K is bounded above by a negative constant

equation $d\varphi = (1-R)\,dA$, one can define a generalized Finsler structure satisfying $K = -1$. I will now describe this construction. For simplicity, I am going to assume that Λ is also time-oriented.

Let $\ell : \Sigma \to \Lambda$ be the bundle of oriented, time oriented g-frames on Λ of the form $(p; e_1, e_2)$ where p is a point of Λ and (e_1, e_2) are an oriented, time oriented basis of $T_p\Lambda$ which satisfies

$$1 = g(e_1, e_1) = -g(e_2, e_2), \qquad 0 = g(e_1, e_2).$$

Then Σ is an $\mathbb{R}$-bundle over Λ and the tautological 1-forms α_1 and α_2 satisfy

$$\begin{aligned} d\alpha_1 &= \psi \wedge \alpha_2 \\ d\alpha_2 &= \psi \wedge \alpha_1 \end{aligned}$$

where ψ is the connection 1-form. It satisfies $d\psi = \ell^* R\,\alpha_1{\wedge}\alpha_2$, where R is the function on Λ which represents the curvature of g.

Now, write $\ell^*\varphi = C\,\alpha_1 - S\,\alpha_2$ for some functions S and C on Σ, which can always be done. Then the hypothesis that $d\varphi = (1-R)\,dA$ ensures that the 1-forms

$$\begin{aligned} \omega &= \psi + \ell^*\varphi = \psi + C\,\alpha_1 - S\,\alpha_2 \\ \theta &= \alpha_2 \\ \eta &= \alpha_1 \end{aligned}$$

satisfy the structure equations of a generalized Finsler structure on Σ with $K \equiv -1$. Further details will be left to the reader.

6.2. A local normal form. In a manner completely analogous to the $K = 1$ case, one can show that there is a local normal form of the form

$$\begin{aligned} \omega &= dz + a\,dx + b\,dy \\ \theta &= \sqrt{b_x - a_y}\big(\sinh z\,dx + \cosh z\,dy\big) \\ \eta &= \sqrt{b_x - a_y}\big(\cosh z\,dx + \sinh z\,dy\big) \end{aligned}$$

where a and b are arbitrary functions of x and y subject to the condition that $b_x - a_y > 0$. Details will be left to the reader.

6.3. Complete examples. To construct complete examples of actual Finsler structures on surfaces, it suffices to construct an example of a Λ endowed with the appropriate structures so that the positive φ-geodesics are are all closed and satisfying appropriate growth conditions.

Again, the method is to start with the above normal form and assume a symmetry. The same sort of analysis that produced the compact examples of Finsler surfaces with $K = -1$ leads to the following prescription: Let Λ be the cylinder $(-\pi/2, \pi/2) \times S^1$ with coordinates ρ and ϕ (which is periodic of period 2π). Let $\Sigma = \Lambda \times \mathbb{R}$ with coordinate z on the $\mathbb{R}$-factor. Let $h : (-\pi/2, \pi/2) \to (-1, 1)$ be an odd function of ρ which satisfies the inequality

$$1 + h(\rho) - h'(\rho)\sin\rho\cos\rho > 0.$$

Now consider the coframing on Σ defined by

$$\begin{aligned}\omega &= dz - (\tan\rho)/\big(1+h(\rho)\big)\,d\phi\\ \theta &= R(\rho)\big(\big(1+h(\rho)\big)\cosh z\,d\rho + \sinh z\,d\phi\big)\\ \eta &= R(\rho)\big(\big(1+h(\rho)\big)\sinh z\,d\rho + \cosh z\,d\phi\big)\end{aligned}$$

where the function $R > 0$ is defined by

$$\big(R(\rho)\big)^2 = \frac{1+h(\rho) - h'(\rho)\sin\rho\cos\rho}{\cos^2\rho\,\big(1+h(\rho)\big)^3}.$$

For any choice of h satisfying the above restrictions, this is a generalized Finsler structure satisfying $K = -1$. A computation shows that the invariant S vanishes if and only if h satisfies the differential equation

$$\frac{1}{\big(1+h(\rho)\big)^2} - \frac{h'(\rho)\tan\rho}{\big(1+h(\rho)\big)^3} = c$$

for some constant c. (Note that $h = 0$ satisfies this condition.)

It is not hard to show that the $\omega\theta$ integral curves are closed in Σ for any such choice of h. (Compare the argument from §5.) They foliate Σ and the leaf projection $\pi : \Sigma \to M$ induces a Finsler structure on M (topologically a disk) which satisfies $K \equiv -1$.

It is not hard to show that if there is a constant $c < 1$ so that $|h(\rho)| \le c$ for all ρ, then the resulting Finsler structure on M is complete. Thus, these choices of h provide examples of complete non-Riemannian Finsler structures on the disk that satisfy $K \equiv -1$.

References

[Ak] H. Akbar-Zadeh, *Sur les espaces de Finsler a courbures sectionnelles constantes*, Acad. Roy. Belg. Bull. Cl. Sci. (5) **74** (1988), 281–322.

[BaChSh] D. Bao, S.-S. Chern, and Z. Shen, *An Introduction to Finsler Geometry*, Springer-Verlag, New York (manuscript in preparation).

[BrChG] R. Bryant, et al, *Exterior Differential Systems*, Springer-Verlag, New York, 1991.

[Be] A. Besse, *Manifolds all of whose geodesics are closed*, Ergeb. Math. Grenzgeb. Band 93, Springer-Verlag, Berlin and New York, 1978.

[Br1] R. Bryant, *Two exotic holonomies in dimension four, path geometries, and twistor theory*, Complex Geometry and Lie Theory, Proceedings of Symposia in Pure Mathematics, vol. 53, American Mathematical Society, Providence, 1991, pp. 33–88.

[Br2] R. Bryant, *Lectures on the Geometry of Partial Differential Equations*, CRM Monograph Series, American Mathematical Society, Providence, 1996 (manuscript in preparation).

[Ca1] É. Cartan, *Les Espace Métrique Fondés sur la Notion d'Aire*, Exposés de Géometrie, t. 72, Hermann, Paris, 1933.

[Ca2] É. Cartan, *Les Espace Finsler*, Exposés de Géometrie, t. 79, Hermann, Paris, 1934.

[Ca3] É. Cartan, *Les espaces généralisés et l'intégration de certaines classes d'équations différentielles*, C. R. Acad. Sc. **206** (1938), 1689–1693.

[Ca4] É. Cartan, *La geometria de las ecuaciones diferenciales de tercer orden*, Revista. Mat. Hispano-Amer. **4** (1941), 1–31.

[Ca5] É. Cartan, *Sur une classe d'espaces de Weyl*, Annales École Normale **60** (1943), 1–16.

[Ch1] S.S. Chern, *On the Euclidean connection in a Finsler space*, Proc. Natl. Acad. Sci. USA **29** (1943), 33–37 (Reprinted in *Shiing-shen Chern: Selected Papers, vol. II*, Springer-Verlag, New York, 1989, pp. 107–111).

[Ch2] S.S. Chern, *Local equivalence and Euclidean connections in Finsler spaces*, Science Reports Tsing Hua Univ. **5** (1948), 95–121 (Reprinted in *Shiing-shen Chern: Selected Papers, vol. II*, Springer-Verlag, New York, 1989, pp. 194–212.).

[Da] G. Darboux, *Leçons sur la Théorie générale des Surfaces, t. 3*, Editions Chelsea, Third edition, 1972, Gauthier-Villars, Paris, 1894–1915.

[Ga] R. Gardner, *The Method of Equivalence and its Applications*, CBMS-NSF Regional Conference Series in Applied Mathematics **58**, SIAM, Philadelphia, 1989.

[GaWi] R. Gardner and G. Wilkens, *Classical geometries arising in feedback equivalence* (1993), Fields Institute, Waterloo (preprint).

[Ko] S. Kobayashi, *?The group of diffeomorphisms preserving a parallelization?*, ?? **??** (??), ??–??.

Department of Mathematics, Duke University, PO Box 90320, Durham, NC 27708-0320

E-mail address: bryant@math.duke.edu