NOTES ON GEODESICS ON LIE GROUPS

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Abstract. These are my notes on the calculation of geodesics on Lie groups using the geometric Euler-Lagrange formalism.

Contents

1. Introduction

In this section, I will review the geometric formulation of the Euler-Lagrange equations on a manifold.

1.1. **Tangent bundles.** Let M^n be a smooth n-manifold and let TM denote its tangent bundle, with basepoint projection $\pi : TM \to M$. Each fiber of π is a vector space $\pi^{-1}(x) = T_xM$, and, as a consequence, there is a canonical isomorphism

$$T_{\pi(v)}M \to \ker \pi'(v)$$

for each tangent vector $v \in TM$, where $\ker \pi'(v) \subset T_v(TM)$ is the kernel of the surjection $\pi'(v) : T_v(TM) \to T_{\pi(v)}M$. The composition of $\pi'(v)$ with this isomorphism is then a nilpotent endomorphism

$$\alpha_v : T_v(TM) \to T_v(TM)$$

and this vector bundle endomorphism $\alpha : T(TM) \to T(TM)$ of rank n is an important feature of the geometry of TM. It is natural in the sense that it commutes with the induced action on TM of any diffeomorphism of M with itself.

Another natural object on TM is the *radial vector field* R on TM. This is the vector field whose time t flow is scalar multiplication by e^t in the fibers of TM. This will be useful below.

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1.2. **Lagrangians.** A *Lagrangian* is (smooth) function $L : TM \to \mathbb{R}$. Given a differentiable curve $\gamma : [a,b] \to M$, one defines the associated functional

$$\mathcal{F}_L(\gamma) = \int_a^b L\big(\gamma'(t)\big)\ dt.$$

Let $p,q \in M$ be given, and let $\Omega\big([a,b],p,q\big)$ denote the set of differentiable mappings $\gamma : [a,b] \to M$ such that $\gamma(a) = p$ and $\gamma(b) = q$. Then $\mathcal{F}_L$ can be regarded as a function on $\Omega\big([a,b],p,q\big)$ and one is interested in its *critical points*, where a curve $\gamma \in \Omega\big([a,b],p,q\big)$ is *critical* if the restriction of $\mathcal{F}_L$ to any 1-parameter smooth variation of γ within $\Omega\big([a,b],p,q\big)$ has γ as a critical point.

The equations that characterize critical curves of a given Lagrangian can be expressed directly in terms of the geometry of TM.

Given a Lagrangian $L : TM \to \mathbb{R}$, one can define a canonical 1-form

$$\omega_L = \mathrm{d}L \circ \alpha$$

on TM. One says that L is *nondegenerate* if the 2-form $\mathrm{d}\omega_L$ is nondegenerate on TM.

One can also define the associated *energy function* of L, which is the function $E_L : TM \to \mathbb{R}$ defined by

$$E_L = R(L) - L.$$

Remark 1. If $L : TM \to \mathbb{R}$ is a Lagrangian that is a homogeneous quadratic polynomial on each tangent space T_xM (as would be the case for the action Lagrangian of a pseudo-Riemannian metric on M), then, by Euler's Theorem, one has $E_L = L$, which should be born in mind for later purposes.

Using these quantities, one has the following classical result, which is a formulation of the Euler-Lagrange equations for a nondegenerate Lagrangian. For a proof (and a discussion of the notation introduced above), the reader might consult Lecture 4 in [1].

Theorem 1 (Euler-Lagrange)**.** *Let $L : TM \to \mathbb{R}$ be a nondegenerate Lagrangian, and let X_L be the unique vector field on TM that satisfies*

$$X_L \lrcorner\, \mathrm{d}\omega_L = -\mathrm{d}E_L\,. \tag{1.1}$$

If $\gamma : [a,b] \to M$ is a critical curve for the functional $\mathcal{F}_L$ on the set $\Omega\big([a,b],\gamma(a),\gamma(b)\big)$, then $\gamma' : [a,b] \to TM$ is an integral curve of X_L. Conversely, every integral curve of X_L, say $\phi : [a,b] \to TM$, is of the form $\phi = \gamma'$ where the curve $\gamma = \pi \circ \phi$ is a critical curve for the functional $\mathcal{F}_L$ on the set $\Omega\big([a,b],\gamma(a),\gamma(b)\big)$.

Thus, finding the critical curves of the functional $\mathcal{F}_L$ under fixed-endpoint variations is equivalent to finding the integral curves of the vector field X_L, which is called the *Euler-Lagrange vector field* of the nondegenerate Lagrangian L.

2. Left invariant quadratic Lagrangians

This section is an elaboration of Exercises 10 and 11 in Lecture 7 of [1].

2.1. **Lie groups and Lie algebras.** Now let G be a Lie group, with Lie algebra $\mathfrak{g}$. (For simplicity, one could keep in mind the case $G = \mathrm{GL}(n,\mathbb{R})$, in which case $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{R})$ is the vector space $M_n(\mathbb{R})$ of n-by-n matrices with real entries.)

Let ζ be the canonical left-invariant $\mathfrak{g}$-valued 1-form on G. Then, as usual, $\mathrm{d}\zeta = -\frac{1}{2}\,[\zeta,\zeta]$. (In the case $G = \mathrm{GL}(n,\mathbb{R})$ and $g : \mathrm{GL}(n,\mathbb{R}) \hookrightarrow M_n(\mathbb{R})$ is the (vector-valued) inclusion mapping, then $\zeta = g^{-1}\,\mathrm{d}g$. Moreover, in this case, one has the more explicit formula $\mathrm{d}\zeta = -\zeta{\wedge}\zeta$.)

It is important to recognize that ζ can be thought of both as a 1-form on G *and* as a function on TG (with values in $\mathfrak{g}$, of course).

To avoid confusion, I will write $z : TG \to \mathfrak{g}$ to denote the *function* on TG that ζ represents. It is not hard to show that one has the identity

$$\pi^*\zeta = \mathrm{d}z \circ \alpha, \tag{2.1}$$

as 1-forms on TG, and this identity will be important in what follows.

2.2. **Left-invariant Lagrangians.** Now, let $Q : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ be a nondegenerate quadratic form on $\mathfrak{g}$, and define the Lagrangian $L : TG \to \mathbb{R}$

$$L = \tfrac{1}{2}\,Q(z,z).$$

Since L is homogeneous quadratic on each fiber of $\pi : TG \to G$, it follows that $E_L = L$.

Now, by the above formula

$$\omega_L = \mathrm{d}L \circ \alpha = Q(z,\mathrm{d}z)\circ\alpha = Q(z,\mathrm{d}z\circ\alpha) = Q\big(z,\pi^*\zeta\big).$$

From this, one computes

$$\mathrm{d}\omega_L = Q(\mathrm{d}z,\pi^*\zeta) - \tfrac{1}{2}\,Q\big(z,[\pi^*\zeta,\pi^*\zeta]\big).$$

Since the $\mathfrak{g}\oplus\mathfrak{g}$-valued form $\big(\pi^*\zeta,\mathrm{d}z\big)$ defines a coframing on TG, if follows from the nondegeneracy of Q that $\mathrm{d}\omega_L$ is nondegenerate as well, so that L is a nondegenerate Lagrangian.

Let X_L be the Euler-Lagrange vector field and define $v : TG \to \mathfrak{g}$ and $a : TG \to \mathfrak{g}$ so that $\pi^*\zeta(X_L) = v$ and $\mathrm{d}z(X_L) = a$. Then one computes that

$$X_L \lrcorner\, \mathrm{d}\omega_L = Q(a,\pi^*\zeta) - Q(\mathrm{d}z,v) - Q\big(z,[v,\pi^*\zeta]\big).$$

Meanwhile, $-\mathrm{d}E_L = -\mathrm{d}L = -Q(z,\mathrm{d}z)$. Comparing coefficients of $\mathrm{d}z$ in the equation $X_L \lrcorner\, \mathrm{d}\omega_L = -\mathrm{d}E_L$, one sees that

$$v = z$$

and that, consequently, a must satisfy the equation

$$Q(a,\pi^*\zeta) = Q\big(z,[v,\pi^*\zeta]\big) = Q\big(z,[z,\pi^*\zeta]\big) = Q\big(z,\mathrm{ad}(z)(\pi^*\zeta)\big) = Q\big(\mathrm{ad}^*_Q(z)z,\pi^*\zeta\big),$$

where, for $p \in \mathfrak{g}$, the linear map $\mathrm{ad}^*_Q(p) : \mathfrak{g} \to \mathfrak{g}$ is the adjoint with respect to the quadratic form Q of the linear map $\mathrm{ad}(p) : \mathfrak{g} \to \mathfrak{g}$. Thus, by the nondegeneracy of Q, one see that one must have

$$a = \mathrm{ad}^*_Q(z)z.$$

Thus, X_L satisfies

$$\pi^*(\zeta)(X_L) = z \qquad \text{and} \qquad \mathrm{d}z(X_L) = \mathrm{ad}^*_Q(z)z, \tag{2.2}$$

which determines X_L uniquely.

Now, to find the L-geodesics, it suffices to find the integral curves of X_L. It is worthwhile noting that (2.2) can be integrated in two stages: First, one finds the integral curves $z : [a,b] \to \mathfrak{g}$ of the ordinary differential equation

$$\dot{z} = \operatorname{ad}^*_Q(z)z, \tag{2.3}$$

which is known as the *Euler equation* of the Lagrangian. Then, for each such solution $z : [a,b] \to \mathfrak{g}$, one solves the left-invariant ordinary differential equation for $g : [a,b] \to G$

$$\zeta\bigl(\dot{\gamma}(t)\bigr) = z(t). \tag{2.4}$$

This gives the L-critical curves on G, i.e., the curves that are the geodesics for the left-invariant pseudo-Riemannian metric $\mathrm{d}s^2 = Q(\zeta,\zeta)$ on G.

Example 1 (Biïnvariant metrics). Suppose that $Q : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ is $\operatorname{Ad}(G)$-invariant and nondegenerate, so that

$$Q(x,[y,z]) + Q([y,x],z) = 0.$$

In this case, $\operatorname{ad}^*_Q(y) = -\operatorname{ad}(y)$ for all $y \in \mathfrak{g}$, so that (2.3) simplifies to

$$\dot{z} = \operatorname{ad}^*_Q(z)z = -\operatorname{ad}(z)z = -[z,z] = 0,$$

so the solutions of the Euler Equations are simply to have z be a constant z_0. Then the remaining equation (2.4) becomes

$$\zeta\bigl(\dot{\gamma}(t)\bigr) = z_0$$

and the general solution of this equation is, of course

$$g(t) = g_0\ \mathrm{e}^{tz_0},$$

so that the geodesics are the (left) translates of the 1-parameter subgroups of G.

Example 2 (K-biïnvariant metrics). Suppose that G is a connected simple Lie group with maximal compact subgroup $K \subset G$. As is well-known, there exists an involutive automorphism $\sigma : G \to G$ such that K is the fixed subgroup of σ.

For example, suppose that $G = \mathrm{SL}(n,\mathbb{R})$, with $K = \mathrm{SO}(n)$. The involutive automorphism in this case is $\sigma(g) = (g^T)^{-1}$.

Let $\sigma' : \mathfrak{g} \to \mathfrak{g}$ denote the map induced on the Lie algebra by σ and write $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m}$, where $\mathfrak{k} \subset \mathfrak{g}$ is the Lie algebra of K and $\mathfrak{m} \subset \mathfrak{g}$ is the orthogonal complement to $\mathfrak{k}$ under the Killing form $B : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$. Then σ' acts as -1 times the identity on $\mathfrak{m}$. For any element $x \in \mathfrak{g}$, write $x = x_0 + x_1$, where x_0 lies in $\mathfrak{k}$ and x_1 lies in $\mathfrak{m}$. Thus, $\sigma'(x) = x_0 - x_1$.

Continuing with the illustrative example, if $(G,K) = \bigl(\mathrm{GL}(n,\mathbb{R}),\mathrm{SO}(n)\bigr)$, then $\mathfrak{k}$ is the space of skew-symmetric n-by-n matrices while $\mathfrak{m}$ is the space of traceless, symmetric n-by-n matrices.

Now, let c be a fixed nonzero constant and consider the quadratic form $Q : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ defined by

$$Q(x_0+x_1, x_0+x_1) = B(x_1,x_1) - c\,B(x_0,x_0)$$

Because of the usual sign convention for the Killing form B, it is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{m}$, so Q is positive definite on $\mathfrak{m}$ and is positive definite on $\mathfrak{k}$ if and only if $c > 0$. It is nondegenerate as long as $c \neq 0$.

Now, Q is $\operatorname{Ad}(K)$-invariant, but it is not $\operatorname{Ad}(G)$-invariant unless $c = -1$. In particular, the associated Lagrangian $L : TG \to \mathbb{R}$ is left-invariant under the action of G and right-invariant under the action of K.

In this case, one finds that (2.3) becomes

$$\dot{z}_0 + \dot{z}_1 = -(1{+}c)\ [z_0, z_1].$$

Writing $\lambda = (1{+}c)$ for simplicity of notation, the solutions of this Euler system can be written as

$$z = z_0 + z_1 = v_0 + \mathrm{Ad}\big(\mathrm{e}^{-\lambda v_0 t}\big)(v_1),$$

where $v = v_0 + v_1$ is a constant in $\mathfrak{g}$. Thus, the geodesic equation (2.4) becomes

$$\zeta\big(\dot\gamma(t)\big) = v_0 + \mathrm{Ad}\big(\mathrm{e}^{-\lambda v_0 t}\big)(v_1).$$

Writing $\gamma(t) = s(t)\ \mathrm{e}^{\lambda v_0 t}$ for some curve s in G, this becomes

$$\zeta\big(\dot s(t)\big) = (1{-}\lambda)v_0 + v_1\ ,$$

so this is solved by $s(t) = s_0\,\mathrm{e}^{\big((1-\lambda)v_0+v_1\big)t}$ where $s_0 \in G$ is an arbitrary constant.

Thus, finally, one has the equation for geodesics in this metric:

$$\gamma(t) = s_0\,\mathrm{e}^{\big(v_1+(1-\lambda)v_0\big)t}\mathrm{e}^{\lambda v_0 t}.$$

When $s_0 = e$, note that $\gamma'(0) = v_0 + v_1 = v$, so this is the geodesic leaving the identity with initial velocity v.

Note that $c = -1$ implies $\lambda = 0$, so this reproduces the case of a biïnvariant metric on a simple Lie group already done in the first example.

On the other hand, when $c = 1$, one has $\lambda = 2$, and this gives the geodesics of a positive definite (Riemannian) metric on G that is K-invariant in the expected form

$$\gamma(t) = s_0\,\mathrm{e}^{(v_1-v_0)t}\,\mathrm{e}^{2v_0 t}.$$

Note that, in the case of $G = \mathrm{SL}(n,\mathbb{R})$, this becomes the formula

$$\gamma(t) = \mathrm{e}^{v^T t}\ \mathrm{e}^{(v-v^T)t}$$

for the geodesic starting at the origin with initial velocity $v \in \mathfrak{g}$.

References

[1] R. Bryant, *An introduction to Lie Groups and symplectic geometry*, AMS, 1991. 2

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