ON THE EQUATION $f''(x) = f\big(f(x)\big)$

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Abstract. These are my notes on the MathOverflow Question "Is there a general solution for the differential equation $f''(x) = f\big(f(x)\big)$", which is MO Question 384174.

1. Rough Solutions

There are many local solutions of this equation. For example, suppose that one starts with a C^2 function f on an interval $I \subset \mathbb{R}$ such that f' is positive on I and $f(I)$ is disjoint from I. Then an inverse $g : f(I) \to I$ of $f : I \to f(I)$ exists and is C^2. Now define f on the interval $f(I)$ so that $f(y) = f''\big(g(y)\big)$ for $y \in f(I)$. Then for $x \in I$, one will have $x = g(y)$ for some $y \in f(I)$ and, of course, $y = f(x)$. Then $f''(x) = f''\big(g(y)\big) = f(y) = f\big(f(x)\big)$ for all $x \in I$. This sort of 'rough' solution is constructed without any fixed points.

A C^2 solution on an open domain D such that $f(D) \subset D$ must be smooth on D, since $f'' = f\circ f$, implying that if f is C^k, then f must be C^{k+2}.

In fact, with a little effort, one can show that a C^2 solution with a contracting fixed point, i.e., a point $x \in \mathrm{dom}(f)$ with $f(a) = a$ and $|f'(a)| < 1$, must be real-analytic in a neighborhood of the fixed point a, since the equation $f'' = f\circ f$ allows one to prove an estimate of the form $|f^{(k)}| \le C^k\, k!$ for some constant C on a neighborhood of such a fixed point.

2. Formal power series

For the investigation of the solutions of $f'' = f\circ f$ that have fixed points, one can begin with 'formal solutions'. It is simplest to start by assuming that $z = 0$ is a fixed point, and to allow complex coefficients (though we are mainly interested in the real case).

Proposition 1: For every constant $b \in \mathbb{C}$, there is a (unique) formal

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power series with lowest order term bz of the form

$$f(z) = bz + \frac{b^2}{3!}\,z^3 + \frac{b^3\,(b^2{+}1)}{5!}\,z^5 + \frac{b^4\,(b^6{+}b^4{+}11\,b^2{+}1)}{7!}\,z^7 + \cdots \tag{1}$$

that satisfies $f''(z) = f(f(z))$.

Proof: Recall that, if

$$f(z) = a_1\,z + a_2\,z^2 + a_3\,z^3 + \cdots$$

is a formal power series (note that the constant term is assumed to be zero), then the formal composition f with itself is a formal power series

$$f{\circ}f(z) = a_1^2\,z + a_1(a_1{+}1)a_2\,z^2 + \cdots + s_n(a)\,z^n + \cdots$$

where $s_k(a)$ is a polynomial with nonnegative integer coefficients given by the classical formula

$$s_n(a) = \sum_{k=1}^{\infty}\ \sum_{j_1+\cdots j_k=n} a_k a_{j_1}\cdots a_{j_k}\,.$$

Note that $s_n(a)$ depends only on $a_1,\dots,a_k$, and, moreover, since any partition of an even integer into an odd number of (integer) parts always has at least one even part, it follows that $s_{2n}(a)$ lies in the ideal generated by $\{a_{2j}\mid 1\le j\le n\}$.

Since the formal derivative satisfies

$$f''(z) = 2a_2 + 6a_3\,z + \cdots + (k{+}2)(k{+}1)a_{k+2}\,z^k + \cdots\,,$$

it follows that $f'' = f{\circ}f$ if and only if

$$(k{+}2)(k{+}1)a_{k+2} = s_k(a_1,\dots,a_k)$$

for all $k\ge 0$. This gives $a_2 = 0$ and, by recursion, one obtains formulae

$$a_{2k} = 0 \qquad\text{and}\qquad a_{2k+1} = p_k(a_1), \qquad\text{for } k\ge 0,$$

where p_k is a polynomial of degree $k^2{+}1$ and leading coefficient $1/(2k{+}1)!$. Moreover, one can prove by induction that $(2k{+}1)!\,p_k(b)/b^{k+1} = q_k(b^2)$, where q_k has degree $\binom{k}{2}$ and nonnegative integer coefficients, which will turn out to be useful in the sequel. QED

The first few terms are $p_0(b) = b$, $p_1(b) = \frac{1}{6}\,b^2$,

$$p_2(b) = \tfrac{1}{120}\,b^3(b^2{+}1),\quad p_3(b) = \tfrac{1}{5040}b^4(b^6{+}b^4{+}11b^2{+}1),\quad \cdots\,.$$

The above proposition has a generalization to power series centered at $z = a$ for any a. The proof is by the same method as above.

Proposition 2: For every pair of constants $a, b \in \mathbb{C}$, there is a (unique) formal power series centered at $z = a$ of the form

$$f(z) = a + b\,(z{-}a) + \cdots = \sum_{k=0}^{\infty} p_k(a,b)\,(z{-}a)^k, \tag{2}$$

that satisfies $f''(z) = f(f(z))$. For each $k \geq 0$, $p_k(a,b)$ is a polynomial in a and b with nonnegative coefficients.

The first few terms are given by

$$\begin{aligned} f(z) = a + b\,(z{-}a) + \tfrac{1}{2}a\,(z{-}a)^2 + \tfrac{1}{6}b^2\,(z{-}a)^3 \\ + \tfrac{1}{24}ab(b{+}1)\,(z{-}a)^4 + \tfrac{1}{120}b(b^4{+}b^2{+}3a^2)\,(z{-}a)^5 + \cdots . \end{aligned} \tag{3}$$

One says that f has a formal fixed point at $z = a$ and that this fixed point is *formally contracting* if $|b| < 1$ and *formally expanding* if $|b| > 1$. If the power series does have a positive radius of convergence, then a is actually a fixed point of limiting holomorphic mapping and actually is contracting if $|b| < 1$ and expanding if $|b| > 1$.

Remark: The formal power series does not always have a positive radius of convergence. For example, when a and b are real and nonnegative,

$$p_{2k+1}(a,b) \geq p_{2k+1}(0,b) = p_k(b) \geq \frac{b^{k^2+1}}{(2k{+}1)!}.$$

Thus, when $a \geq 0$ and $b > 1$, the sequence $p_{2k+1}(a,b)$ grows faster than any geometric sequence C^k and hence the general term of the series does not approach zero for any $z \neq a$. (This simple argument was pointed out to me by Will Sawin.)

3. Convergence near a formal contracting fixed point

Now, suppose that $a = 0$ and $0 < b < 1$ and that one has a function f defined on an open neighborhood of $x = 0$ that satisfies $f(0) = 0$, $f'(0) = b > 0$ and $f'' = f{\circ}f$. Since $0 < f'(0) < 1$, there will be some $\delta > 0$ such that $0 < b/2 < f'(x) < \frac{1}{2}(1 + b) < 1$ when $|x| < \delta$. Then $f : (-\delta, \delta) \to (-\delta, \delta)$ is a contraction mapping and thus must be smooth on $(-\delta, \delta)$. Moreover, f is strictly increasing on $(-\delta, \delta)$ and satisfies $0 < f(x) < x$ for all $0 < x < \delta$. Integrating the given relation $f'' = f{\circ}f$ twice, one obtains

$$f(x) = bx + \int_0^x \int_0^y f\big(f(t)\big)\,\mathrm{d}t\,\mathrm{d}y.$$

This motivates the following construction and argument.

3.1. **An iterative procedure.** Let b, μ and r be real constants satisfying $0 < b < \mu \le 1$ and $r > 0$. Let $\mathcal{M}(b,\mu,r)$ be the set of continuous, nondecreasing functions u on $[0,r]$ satisfying $u(0) = 0$ and $0 \le u(x) \le \mu x$ for all $x \in [0,r]$. (For example, the function $u(x) = bx$ belongs to M.) For $u \in \mathcal{M}(b,\mu,r)$, define $Su : [0,r] \to \mathbb{R}$ by

$$Su(x) = bx + \int_0^x \int_0^y u\big(u(t)\big)\,\mathrm{d}t\,\mathrm{d}y.$$

Proposition 3: When $r^2 \le 6(\mu{-}b)/\mu^2$, the nonlinear operator S carries $\mathcal{M}(b,\mu,r)$ into itself. Moreover, if $u, v \in \mathcal{M}(b,\mu,r)$ satisfy $u(x) \le v(x)$ for all $x \in [0,r]$, then $Su(x) \le Sv(x)$ for all $x \in [0,r]$.

Proof: Suppose that $u \in \mathcal{M}(b,\mu,r)$. Then Su satisfies $Su(0) = 0$. Since, $0 \le u(x) \le \mu x \le \mu r \le r$ for $x \in [0,r]$, it follows that $u{\circ}u$ is well-defined, continuous, and nonnegative on $[0,r]$. Hence

$$(Su)'(x) = b + \int_0^x u\big(u(t)\big)\,\mathrm{d}t \ge b > 0 \qquad (\text{when } x \in [0,r]),$$

so it follows that Su is nondecreasing on $[0,r]$.

Moreover, if $u \in \mathcal{M}(b,\mu,r)$, then $u(u(t)) \le \mu\, u(t) \le \mu^2\, t$ for all $t \in [0,r]$ so

$$Su(x) = bx + \int_0^x \int_0^y u\big(u(t)\big)\,\mathrm{d}t\,\mathrm{d}y \le bx + \int_0^x \int_0^y \mu^2 t\,\mathrm{d}t\,\mathrm{d}y = bx + \tfrac{1}{6}\,\mu^2\,x^3.$$

Meanwhile, when $x \in [0,r]$,

$$bx + \tfrac{1}{6}\,\mu^2\,x^3 \le bx + \tfrac{1}{6}\,\mu^2\,r^2 x \le bx + (\mu - b)x = \mu x.$$

Thus, one has $0 \le Su(x) \le \mu x$ when $x \in [0,r]$. It follows that $Su \in \mathcal{M}(b,\mu,r)$.

Finally, if $u, v \in \mathcal{M}(b,\mu,r)$ satisfy $u(x) \le v(x)$ for all $x \in [0,r]$ then $u(u(t)) \le u(v(t)) \le v(v(t))$, which implies that $Su(x) \le Sv(x)$ for $x \in [0,r]$. QED

Proposition 4: When b, μ, and $r > 0$ satisfy $0 < b < \mu \le 1$ and $r^2 \le 6(\mu{-}b)/\mu^2$, the power series $f(x) = \sum_{k=0}^{\infty} p_k(b)\,x^{2k+1}$ converges for $x \in [0,r]$, and the limiting function satisfies $0 \le f(x) \le \mu x$ for $x \in [0,r]$.

Proof: Let $u_0(x) = bx$, and set $u_{i+1} = Su_i$ for $i \ge 0$. Since $u_0 \in \mathcal{M}(b,\mu,r)$, it follows that $u_i \in \mathcal{M}(b,\mu,r)$ for all $i \ge 0$. Also, when $x \in [0,r]$, since $u_0(x) = bx \le bx + \frac{1}{6}\,b^2\,x^3 = u_1(x)$, it follows that $u_1(x) = Su_0(x) \le Su_1(x) = u_2(x)$ and, by induction, $u_i(x) \le u_{i+1}(x) \le \mu\,x$ for all i and $x \in [0,r]$.

Now, direct computation gives

$$u_2(x) = bx + \tfrac{1}{6}\, b^2\, x^3 + \tfrac{1}{5!}\, b^3(b^2+1)\, x^5 + R_7(x),$$

where $R_7(x)$ is a polynomial in x that vanishes to order 7 at $x = 0$ and all of its coefficients are nonnegative. Moreover, by induction, one can show that

$$u_k(x) = b\, x + p_1(b)\, x^3 + \cdots + p_k(b)\, x^{2k+1} + R_{2k+3}(x)$$

where $R_{2k+3}(x)$ is a polynomial in x that vanishes to order $2k{+}3$ at $x = 0$ and all of its coefficients are nonnegative. Hence, for all k and $x \in [0, r]$,

$$bx + p_1(b)\, x^3 + \cdots + p_k(b)\, x^{2k+1} < u_k(x) \le \mu\, x.$$

Thus, when $0 < b < 1$ and $x \in [0, r]$, we have

$$\sum_{k=0}^{\infty} p_k(b)\, x^{2k+1} \le \mu\, x.$$

This completes the proof of the Proposition. QED

Corollary: When $0 < b < 1$, the formal power series (1) has a positive radius of convergence, say, $r(b) > 0$.

One can get a lower bound for $r(b)$ as follows: We know that $r(b)^2 \ge 6(\mu - b)/\mu^2 > 0$ for all $\mu \in (b, 1]$, so it suffices to pick $\mu \in (b, 1]$ to maximize $(\mu - b)/\mu^2$. When $0 < b \le \frac{1}{2}$, the maximum of $(\mu - b)/\mu^2$ on $(b, 1]$, occurs at $\mu = 2b$, and when $\frac{1}{2} \le b < 1$ the maximum occurs at $\mu = 1$. This yields the (non-sharp) lower bound

$$r(b) \ge \begin{cases} \sqrt{\dfrac{3}{2b}} & 0 < b \le \frac{1}{2}, \\[2ex] \sqrt{6(1-b)} & \frac{1}{2} \le b < 1. \end{cases}$$

The convergence result in the real case generalizes to the complex case as follows:

Proposition 5: When $|b| < 1$, the power series $f(z) = \sum_{k=0}^{\infty} p_k(b)\, z^{2k+1}$ converges absolutely and uniformly on the disk $|z| \le r\big(|b|\big)$, and the limiting holomorphic function satisfies $|f(z)| \le |z|$ for $|z| \le r\big(|b|\big)$.

Proof: If $b \in \mathbb{C}$ satisfies $|b| < 1$, then the fact that the coefficients of $p_k(b)$ are nonnegative real numbers implies that $|p_k(b)| \le p_k(|b|)$.

Hence, when $z \in \mathbb{C}$ satisfies $|z| \le r\big(|b|\big)$, we have

$$\left|\sum_{k=0}^{\infty} p_k(b)\, z^{2k+1}\right| \le \sum_{k=0}^{\infty} p_k(|b|)\, |z|^{2k+1} \le |z|.$$

This completes the proof. QED

There is also a (non-sharp) upper bound for $r(b)$:

Proposition 6: When $0 < b < 1$, $r(b) > 0$ is finite and satisfies

$$r(b) < \frac{\sqrt{6(1-b)}}{b}.$$

Moreover, when $a \ge 0$ and $b = 1$, the series (2) has zero radius of convergence.

Proof: Suppose that $r(b) \ge \frac{\sqrt{6(1-b)}}{b}$. Because all of the coefficients of the formal power series are non-negative, it follows that, for $0 \le x \le r(b)$, we have $f(x) > bx + \frac{1}{6}b^2x^3$. Now, when $x = \frac{\sqrt{6(1-b)}}{b}$, we have $bx + \frac{1}{6}b^2x^3 = x$, so $f(x) > x$, implying that there must be a positive $a < \frac{\sqrt{6(1-b)}}{b}$ such that $f(a) = a$. By the mean value theorem, there will exist a c between 0 and a such that $f'(c) = 1$. Since $f'' = f\circ f > 0$ on $\big(0, r(b)\big)$, it follows that $f'(a) > 1$. Thus, f is real-analytic at a and has $f'(a) > 1$, and so its Taylor series at $x = a$ would have a positive radius of convergence. However, we already know that this is impossible. Thus, $r(b) < \frac{\sqrt{6(1-b)}}{b}$.

Finally, if the series (2) with $a \ge 0$ and $b = 1$ had a positive radius of convergence $\rho > 0$, then, since $0 < p_k(b) < p_k(1) \le p_{2k+1}(a, 1)$ when $0 < b < 1$, the series (1) with $b \in (0, 1)$ would have radius of convergence at least ρ, but we have shown that $r(b) \to 0$ as $b \to 1$. QED

Remark: A similar argument using $f(x) > bx + \frac{1}{6}b^2x^3 + \frac{1}{120}b^3(b^2+1)x^5$ for $0 < x < r(b)$ gives a lower (but more messy) upper bound

$$r(b) < \frac{1}{b^{3/4}}\sqrt{\frac{30(1-b)}{\sqrt{30-5b+30b^2-30b^3} + 5\sqrt{b}}}\,,$$

which is a significantly better bound when $b > 0$ is small than the estimate proved in the proposition, though it is quite close to the simpler upper bound when $b < 1$ is close to 1.

Proposition 7: For $0 < b < 1$, the series (1), with radius of convergence $r(b) > 0$ is the Taylor series at $x = 0$ of an odd real-analytic

function $f : \big(-r(b), r(b)\big) \to \big(-r(b), r(b)\big)$ that is strictly increasing and satisfies $\lim_{x\to r(b)^-} f(x) = r(b)$. The function f satisfies $f'' = f\circ f$ and cannot be extended real-analytically to any larger interval.

Proof: Let $0 < b < 1$ and let $R \ge r(b)$ be the largest real number (or ∞) such that the real-analytic function f to which the series (1) converges on $\big(-r(b), r(b)\big)$ can be extended real-analytically to $(-R, R)$. Note that, because $|f'(0)| < 1$, $x = 0$ is a contracting fixed point of f. Hence there is an interval $(-\epsilon, \epsilon)$ for some $\epsilon > 0$ that is preserved by f. In particular, $f\circ f$ is defined and real-analytic on this interval and hence, by construction, it equals f'' on this interval.

N.B.: I am *not* assuming that the series (1) converges on $(-R, R)$, nor am I assuming that f maps $(-R, R)$ to itself. In particular, while f'' is defined on $(-R, R)$, it is not immediate that $f\circ f$ is defined on $(-R, R)$ since we do not know that f carries $(-R, R)$ into itself. This must be addressed.

First, I claim that $f(x) < x$ for $0 < x < R$. Suppose not. Since $f(x) < x$ for $0 < x < \epsilon$, there must be a smallest real number $a \in (0, R)$ such that $f(a) = a$, in particular $f(x) < x$ for $0 < x < a$. Then $f'(a) \ge 1$, otherwise, $f(x) - x$ would have negative derivative at $x = a$ and hence would be positive for some $a' \in (0, a)$, a contradiction. However, since f is real-analytic at a, this would imply that the Taylor series of f at $x = a$ has positive radius of convergence, contradicting the result above that says that the series (2) with $(a, b) = \big(a, f'(a)\big)$ has radius of convergence 0.

Second, I claim that f is positive on $(0, R)$. Suppose not. Then there would be a smallest positive $a_0 \in (0, R)$ such that $f(a_0) = 0$. By the mean value theorem, there exists an $a_1 \in (0, a_0)$ such that $f'(a_1) = 0$, which can be supposed to be the smallest positive root of f'. Then f is strictly increasing on $[0, a_1]$, so that $0 < f(x) < x$ for all $x \in (0, a_1)$ and, by the identity $f(-x) = -x$, we have $x < f(x) < 0$ for $x \in (-a_1, 0)$. Consequently, f maps $(0, a_1)$ into $(0, a_1)$. Thus, $f\circ f$ is well-defined and real-analytic on $(-a_1, a_1)$ and hence must equal f'' on this interval. Hence, f'' is positive on $(0, a_1)$. Hence, f' is increasing on $[0, a_1]$ However, since $f'(0) = b > 0$ but $f'(a_1) = 0$, this is impossible.

Thus, f must satisfy $0 < f(x) < x$ when $0 < x < R$, implying (since f is odd) that f maps $(-R, R)$ into $(-R, R)$. Consequently, $f\circ f$ is well-defined on $(-R, R)$ and real-analytic, so it must equal f'' on $(-R, R)$. In particular, since f carries $(0, R)$ into itself, it follows that $f'' = f\circ f$ must also carry $(0, R)$ into itself. Thus, $f' \ge b > 0$ is strictly increasing on $[0, R)$.

In particular, f is increasing and convex up on $[0,R)$. In fact, since $f'>b$ on $(0,R)$, it follows that $f(t)>bt$ and $f''(t)=f(f(t))>b^2t$ for $t\in(0,R)$. Hence

$$f(x) = bx + \int_0^x\int_0^y f(f(t))\,dt,dy > bx + \tfrac16 b^2x^3$$

for $x\in(0,R)$. Since $f(x)<x$ for $x\in(0,R)$, it follows that $bx+\frac16b^2x^3<x$ for $x\in(0,R)$. Thus, $x<(\sqrt{6(1-b)})/b$, implying that $R\le(\sqrt{6(1-b)})/b$.

Because f' is even and increasing on $(0,R)$ it follows that $f'\ge b$ on $(-R,R)$. Thus, f maps $(-R,R)$ diffeomorphically onto $(-S,S)$ for some $S\le R$, and hence is invertible as a real-analytic mapping. I will now show that $S=R$, i.e., that for every $x\in(0,R)$ there exists a $y\in(0,R)$ such that $f(y)=x$.

To do this, let $F(x)=\int_0^x f(t)\,dt$ be the anti-derivative of f that satisfies $F(0)=0$. Then $F(-x)=F(x)$ and F is defined and real-analytic on $(-R,R)$ and has a strict minimum at $x=0$. Moreover,

$$f''(x)f'(x) = f(f(x))\,f'(x) = F'\big(f(x)\big)\,f'(x),$$

and integrating this equation from 0 to x yields

$$\tfrac12\big(f'(x)^2-b^2\big) = F\big(f(x)\big),$$

which can be rearranged to give

$$\frac{f'(x)}{\sqrt{b^2+2F\big(f(x)\big)}} = 1.$$

Since F is defined and nonnegative on $(-R,R)$, there is a well-defined function h on $(-R,R)$ such that

$$h(y) = \int_0^y \frac{du}{\sqrt{b^2+2F(u)}}.$$

Note that h is odd, real-analytic, and invertible. Hence there is a $T>0$ such that $h:(-R,R)\to(-T,T)$ is a real-analytic diffeomorphism. Moreover, for $x\in(-R,R)$,

$$h\big(f(x)\big) = \int_0^{f(x)} \frac{du}{\sqrt{b^2+2F(u)}} = \int_0^x \frac{f'(t)\,dt}{\sqrt{b^2+2F\big(f(t)\big)}} = \int_0^x 1dt = x.$$

In other words, $h\circ f:(-R,R)\to(-T,T)$ is the identity on $(-R,R)$. This implies, first, that $T\ge R$. However, if $T>R$, then $h^{-1}:(-T,T)\to(-R,R)$ is a real-analytic extension of $f:(-R,R)\to$

$(-R, R)$, contrary to the assumption that $(-R, R)$ is the maximal domain to which f can be continued real-analytically. Thus, $T = R$, and we see that $f : (-R, R) \to (-R, R)$ is a diffeomorphism, as claimed. Note that we can extend f continuously to $[-R, R]$ by setting $f(\pm R) = \pm R$, so we do this. (In fact, this extension is smooth in the sense that f and all of its derivatives extend continuously to $x = \pm R$, but we do not need this.)

We now have that, for this extended f, which belongs to $\mathcal{M}\big(b, 1, R\big)$,

$$f(x) = bx + \int_0^x \int_0^y f(f(t))\, dt\, dy = Sf(x)$$

for $x \in [0, r(b)]$, and we have that $f(x) \geq u_1(x) = bx$ for $x \in [0, R]$, which, after setting setting $u_{k+1} = Su_k$, implies by induction that $f = Sf \geq Su_k = u_{k+1}$ for all k. Thus, when $x \in [0, R]$, we have

$$x \geq f(x) \geq u_k(x) > \sum_{k=0}^{k} p_i(b) x^{2i+1}.$$

Since all the terms on the righthand sum are positive, it follows that

$$\sum_{k=0}^{\infty} p_i(b) x^{2i+1} \leq f(x) \leq x$$

when $x \in [0, R]$, so that the series (1) defining f converges (absolutely) on $[-R, R]$. Hence, $R \leq r(b)$. In particular, for $|x| \leq R$, we have

$$f(x) = \sum_{k=0}^{\infty} p_i(b) x^{2i+1}$$

Since $R \geq r(b)$ by definition, it follows that $R = r(b)$. Q.E.D.

Remark: It appears that, for b less than 1 but to close to 1, there is a convergent series expansion

$$\frac{r(b)}{\sqrt{6(1-b)}} = 1 + \frac{7}{10}(1-b) + \frac{243}{200}(1-b)^2 + \cdots .$$

Remark: As in the case $a = 0$, when $|b| < 1$, so that f is a 'formal contraction' on a neighborhood of a, it turns out that the series (2) converges absolutely and uniformly on a disc of the form $|z-a| \leq r(a, b)$ for some $r(a, b) > 0$, so this gives a two-parameter family of local solutions with a contracting fixed point.

3.2. **The case $-1 < b < 0$.** It turns out that the behavior of the series f when b is real and lies in $(-1, 0)$ is quite different from the case when b is real and lies in $(0, 1)$.

Proposition 8: When $b = -c^2 < 0$ and c is sufficiently small, the series (1) has a radius of convergence $r(|b|) < \infty$ and converges to a function f on the interval $\big(-r(|b|), r(|b|)\big)$ that extends real-analytically and periodically to the entire line $\mathbb{R}$.

Proof: Here is the idea of the proof: Let f be the odd function to which the series (1) converges on $\big(-r(|b|), r(|b|)\big)$, and let $\rho \le r(|b|)$ be the largest value such that $|f(x)| < r(|b|)$ for $x \in (-\rho, \rho)$. Consider the differential equation with initial conditions for a function $h : \mathbb{R} \to \big(-r(|b|), r(|b|)\big)$ given by

$$h''(x) = f\big(h(x)\big), \qquad h(0) = 0, \quad h'(0) = b.$$

Clearly, $h = f$ on $(-\rho, \rho)$. Our goal is to show that the (unique) solution of the above equation is defined for all $x \in \mathbb{R}$ and is periodic, with $|h(x)| < r(|b|)$ for all x, at least when $b < 0$ is sufficiently small.

To do this, we observe the following: Let $F : \big(-r(|b|), r(|b|)\big) \to \mathbb{R}$ be the antiderivative of f on $\big(-r(|b|), r(|b|)\big)$ that satisfies $F(0) = 0$. Then multiplying both sides of the equation

$$h''(x) = f\big(h(x)\big) = F'\big(h(x)\big)$$

by $h'(x)$ and integrating with respect to x yields

$$\tfrac{1}{2}\big(h'(x)^2 - b^2\big) = F(h(x)).$$

which can be re-arranged to become

$$h'(x)^2 - 2F\big(h(x)\big) - b^2 = 0.$$

Now, consider the power series for the anti-derivative (with respect to z) of f , i.e.,

$$\Phi(b, z) = \sum_{k=0}^{\infty} p_k(b)\,\frac{z^{2k+2}}{(2k+2)} = \sum_{k=0}^{\infty} b^{k+1} q_k(b^2)\,\frac{z^{2k+2}}{(2k+2)!}.$$

The series for $\Phi(b, z)$ is absolutely convergent when $|b| < 1$ and $|z| \le r(|b|)$. The above equation can now be written more explicitly as

$$b^2 - h'(x)^2 + 2\Phi\big(b, h(x)\big) = 0$$

Now, set $b = -c^2$ and consider the function

$$\begin{aligned} Z(c, u, v) &= c^4 - (cv)^2 + 2\Phi(-c^2, u) \\ &= c^4 - (cv)^2 + 2\sum_{k=0}^{\infty} (-c^2)^{k+1} q_k(c^4)\,\frac{(u^2)^{k+1}}{(2k+2)!}, \end{aligned}$$

which is defined and real-analytic on the domain D in cuv-space where $|c| < 1$ and $|u| < r(c^2)$, since this is where the series converges. In particular, this includes the region where $|c| < 1/\sqrt{2}$ and $|cu| < \sqrt{3/2}$. Since every term is divisible by c^2, we can write $Z(c,u,v) = c^2 W(c,u,v)$ where

$$W(c,u,v) = c^2 - v^2 - u^2 - 2u^2 \sum_{k=1}^{\infty} (-c^2)^k q_k(c^4)\, \frac{(u^2)^k}{(2k+2)!},$$

and all the terms in the final sum have degree at least 2 in (c^2, u^2).

It follows that, on an open neighborhood of $(c,u,v) = (0,0,0)$ in D, the locus $W = 0$ can be described by an equation of the form $c^2 = H(u,v)$, where H is an analytic function on a neighborhood of $(0,0)$ that is even in u and v separately. Using the power series for Φ, one can compute the first few terms in the Taylor expansion for H:

$$H(u,v) = (u^2{+}v^2)\left(1 - \frac{u^4}{12} + \frac{u^6(7u^2{+}2v^2)}{720} + \cdots\right)$$

It follows that the function H is strictly convex on a neighborhood of $(u,v) = (0,0)$. Thus, except for the origin itself, an open neighborhood of $(u,v) = (0,0)$ is foliated by the level sets of H, which are closed, strictly convex curves. In particular, there is a $c_0 \in (0,1)$ such that, when $0 < c < c_0$, the component of the level set

$$H(u,v) = c^2$$

that contains $(u,v) = (0,-c)$ is a closed, strictly convex curve X_c surrounding the origin in the uv-plane. Since $H(-u,v) = H(u,-v) = H(u,v)$, the curve X_c is symmetric under reflection in the u- and v-axes. Thus, $|v|$ reaches a nondegenerate maximum on X_c when $u = 0$, and, since $W(c,0,v) = c^2 - v^2$, it follows that the maximum value of $|v|$ on X_c is c itself. Meanwhile, $|u|$ reaches a nondegenerate maximum $\mu(c)$ on X_c when $v = 0$. Since $H\big(\mu(c),0\big) = c^2$, it follows that $W\big(c,\mu(c),0\big) = 0$, and examining the series, $W(c,u,0) = c^2 - u^2 + c^4u^4/12 + \cdots$, one finds that there is a convergent series expansion

$$\mu(c) = c\left(1 + \frac{c^4}{24} + \frac{3\,c^8}{640} - \frac{223\,c^{12}}{322560} + \cdots\right).$$

In particular, note that, on X_c, we have $|cu| \le c\mu(c)$, and, hence, for c sufficiently near 0, we will have $|cu| < \sqrt{3/2}$ on X_c.

Consider the 1-form $\tau = \frac{\mathrm{d}u}{v}$ on X_c. Since u restricted to X_c has critical points only when $v = 0$ and those are nondegenerate, it follows that τ is smooth (in fact, real-analytic) and nowhere vanishing on X_c.

Using the power series for H and μ, it is not difficult to show that

$$\int_{X_c} \tau = \pi(c) = 2\pi\,\bigl(1 + \tfrac{1}{16}\,c^4 + \cdots\bigr).$$

Consequently, there is a $\pi(c)$-periodic parametrization $\psi(t) = \bigl(u(t), v(t)\bigr)$ of X_c with $\psi(0) = (0, -c)$ and $\psi^*(\tau) = \mathrm{d}t$.

By construction, $u'(t) = v(t)$, so that $u(t)$ satisfies the equation $H\bigl(u(t), u'(t)\bigr) = c^2$. Consequently, $u(t)$ satisfies the equation

$$\bigl(c\,u'(t)\bigr)^2 - 2\Phi\bigl(-c^2, u(t)\bigr) = c^4.$$

Thus, setting $t = cx$, one sees that the function h defined for all $x \in \mathbb{R}$ by $h(x) = u(cx)$, which has period $\pi(c)/c = (2\pi/c)(1 + \frac{1}{16}\,c^4 + \cdots)$, satisfies the equation

$$h'(x)^2 - 2\Phi(-c^2, h(x)) = c^4,$$

and the initial conditions $h(0) = 0$ and $h'(0) = -c^2 = b$.

As has been shown, this forces $h = f$ on the interval of convergence of the Taylor series at $x = 0$ of f. Since $h : \mathbb{R} \to \bigl[-\mu(c), \mu(c)\bigr]$ is real-analytic (and periodic), it follows that it is the unique real-analytic extension of f to the entire line $\mathbb{R}$. Since $f'' = f\circ f$ on the domain of convergence, it follows that its real-analytic extension h satisfies $h'' = h\circ h$ on all of $\mathbb{R}$.

Finally, note that, because $\Phi(-b, iz) = \Phi(b, z)$ on the domain of convergence for Φ, it follows that the radius of convergence for the z-power series $\Phi(-b, z)$ is the same as the radius of convergence of the z-power series $\Phi(b, z)$. Hence the radius of convergence of the power series (1) for a given value of b is the same as the radius of convergence of the power series (1) using the value $-b$. QED

Remark: Although the argument above only gives periodicity of the real-analytic extension of f when $b = -c^2$ where c is sufficiently small, numerical evidence seems to indicate that, in fact, the sum of the series (1) extends to a periodic function on $\mathbb{R}$ whenever $0 < c < 1$. This is based on the observation that the plot of the level sets of the truncation to order 12 of the series that defines the function $H(u, v)$ shows them to be convex for all the level values less than or equal to 1. Thus, it seems highly likely to me that the solutions corresponding to $b \in (-1, 0)$ are all periodic.

4. Some explicit solutions

One interesting point is that the (two) multivalued solutions described by Michael Engelhardt have fixed points and hence are (analytic continuations of) solutions of the type (2). One can see this as

follows: These (multivalued) solutions can be written in the form

$$f(x) = i\sqrt{2}\,\left(\frac{x}{i\sqrt{2}}\right)^b, \qquad \text{where } b = \tfrac{1}{2}(1 \pm i\sqrt{7}).$$

Clearly, $a \in \mathbb{C}$ will be a fixed point, i.e., $f(a) = a$ if and only if

$$1 = \left(\frac{a}{i\sqrt{2}}\right)^{b-1},$$

and this happens (for $b = \frac{1}{2}(1 + i\sqrt{7})$) when, for some integer k,

$$a = a_k = i\sqrt{2}\,e^{i\pi k(1+i\sqrt{7})/2} = i^{k+1}\sqrt{2}\,\left(e^{-\pi\sqrt{7}}\right)^{k/2}.$$

Moreover, we have

$$f'(a_k) = b\left(\frac{a_k}{i\sqrt{2}}\right)^{b-1} = b,$$

so $|f'(a_k)| = |b| = \sqrt{2} > 1$, which implies that the fixed point is a repelling fixed point. This is interesting because it implies that the formal power series given above for (a_k, b) must have a positive radius of convergence, even though $|b| > 1$. This led me to speculate that maybe the formal power series (2) might have a positive radius of convergence for any $(a, b) \in \mathbb{C}$, but Will Sawin (in a comment below) pointed out that this cannot be true.

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