Spin(10,1)-metrics with a parallel null spinor and maximal holonomy

0. Introduction. The purpose of this addendum to the earlier notes on spinors is to outline the construction of Lorentzian metrics in 10+1 dimensions that have a parallel null spinor and whose holonomy is as large as possible. The notation from the earlier note will be maintained here.

1. The squaring map. Consider the squaring map $\sigma : \mathbb{O}^4 \to \mathbb{R}^{2+1} \oplus \mathbb{O} = \mathbb{R}^{10+1}$ that takes spinors for $\mathrm{Spin}(10,1)$ to vectors. This map σ is defined as follows:

$$\sigma\left(\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{y}_1 \\ \mathbf{x}_2 \\ \mathbf{y}_2 \end{pmatrix}\right) = \begin{pmatrix} |\mathbf{x}_1|^2 + |\mathbf{y}_1|^2 \\ 2\left(\mathbf{x}_1 \cdot \mathbf{x}_2 - \mathbf{y}_1 \cdot \mathbf{y}_2\right) \\ |\mathbf{x}_2|^2 + |\mathbf{y}_2|^2 \\ 2\left(\mathbf{x}_1\,\mathbf{y}_2 + \mathbf{x}_2\,\mathbf{y}_1\right) \end{pmatrix}.$$

Define the inner product on vectors in $\mathbb{R}^{2+1} \oplus \mathbb{O}$ by the rule

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \mathbf{x} \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \mathbf{y} \end{pmatrix} = -2(a_1 b_3 + a_3 b_1) + a_2 b_2 + \mathbf{x} \cdot \mathbf{y}$$

and let $\mathrm{SO}(10,1)$ denote the subgroup of $\mathrm{SL}(\mathbb{R}^{2+1} \oplus \mathbb{O})$ that preserves this inner product. This group still has two components of course, but only the identity component $\mathrm{SO}^{\uparrow}(10,1)$ will be of interest here. Let $\rho : \mathrm{Spin}(10,1) \to \mathrm{SO}^{\uparrow}(10,1)$ be the homomorphism whose induced map on Lie algebras is given by the isomorphism

$$\rho'\left(\begin{pmatrix} a_1 + x\,I_8 & C\,R_{\mathbf{x}} & y\,I_8 & C\,R_{\mathbf{y}} \\ -C\,L_{\mathbf{x}} & a_3 + x\,I_8 & C\,L_{\mathbf{y}} & -y\,I_8 \\ z\,I_8 & C\,R_{\mathbf{z}} & a_1 - x\,I_8 & C\,R_{\mathbf{x}} \\ C\,L_{\mathbf{z}} & -z\,I_8 & -C\,L_{\mathbf{x}} & a_3 - x\,I_8 \end{pmatrix}\right) = \begin{pmatrix} 2x & y & 0 & \overline{\mathbf{y}}^* \\ 2z & 0 & 2y & 2\,\overline{\mathbf{x}}^* \\ 0 & z & -2x & \overline{\mathbf{z}}^* \\ 2\,\overline{\mathbf{z}} & -2\,\overline{\mathbf{x}} & 2\,\overline{\mathbf{y}} & a_2 \end{pmatrix}.$$

With these definitions, the squaring map σ is seen to have the equivariance $\sigma\big(g\,\mathbf{z}\big) = \rho(g)\,\big(\sigma(\mathbf{z})\big)$ for all g in $\mathrm{Spin}(10,1)$ and all $\mathbf{z} \in \mathbb{O}^4$.

With these definitions, the polynomial p has the expression $p(\mathbf{z}) = -\frac{1}{4}\sigma(\mathbf{z})\cdot\sigma(\mathbf{z})$, from which its invariance is immediate. Moreover, it follows from this that the squaring map carries the orbits of $\mathrm{Spin}(10,1)$ to the orbits of $\mathrm{SO}^{\uparrow}(10,1)$ and that the image of σ is the union of the origin, the forward light cone, and the future-directed timelike vectors.

2. Parallel null spinors. A non-zero spinor $\mathbf{z} \in \mathbb{O}^4$ will be said to be *null* if $p(\mathbf{z}) = 0$, or, equivalently, if $\sigma(\mathbf{z})$ is a null vector in $\mathbb{R}^{10+1}$. The typical example is $\mathbf{z}_{1,0}$, whose stabilizer subgroup H is the connected subgroup of $\mathrm{Spin}(10,1)$ whose Lie algebra is defined by the conditions $x = z = \mathbf{x} = \mathbf{z} = 0$ and $a \in \mathfrak{k}_1$. Consider the subgroup $\rho(H) \subset \mathrm{SO}^{\uparrow}(10,1)$. Its Lie algebra is given by

$$\rho'(\mathfrak{h}) = \left\{ \begin{pmatrix} 0 & y & 0 & \overline{\mathbf{y}}^* \\ 0 & 0 & 2y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\,\overline{\mathbf{y}} & a_2 \end{pmatrix} \;\middle|\; \begin{array}{c} y \in \mathbb{R}, \\ \mathbf{y} \in \mathbb{O}, \\ a \in \mathfrak{k}_1 \end{array} \right\}.$$

The question is whether $\rho(H)$ can be the holonomy of a torsion-free connection on an 11-manifold.

The first thing to check is to see whether this subgroup satisfies the Berger criteria. Suppose that M were an 11-manifold endowed with a $\rho(H)$-structure B that is torsion-free. Then the Cartan structure equations on B will be of the form

$$\begin{pmatrix} d\omega_1 \\ d\omega_2 \\ d\omega_3 \\ d\boldsymbol{\omega} \end{pmatrix} = - \begin{pmatrix} 0 & \psi & 0 & {}^t\boldsymbol{\phi} \\ 0 & 0 & 2\,\psi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\,\boldsymbol{\phi} & \boldsymbol{\theta} \end{pmatrix} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \boldsymbol{\omega} \end{pmatrix}$$

where $\boldsymbol{\omega}$ and $\boldsymbol{\phi}$ take values in $\mathbb{O}$ and $\boldsymbol{\theta}$ takes values in the subalgebra $\mathfrak{spin}(7) \subset \mathfrak{gl}(\mathbb{O})$ that consists of the elements of the form a_2 with $a \in \mathfrak{k}_1$. For such a $\rho(H)$-structure, the Lorentzian metric $g = -4\,\omega_1\,\omega_3 + {\omega_2}^2 + \boldsymbol{\omega}\cdot\boldsymbol{\omega}$ has a parallel null spinor and B represents the structure reduction afforded by this parallel structure. Note that the null 1-form ω_3 is parallel and well-defined on M. It (or, more properly, its metric dual vector field) is the square of the parallel null spinor field.

Differentiating the Cartan structure equations yields the first Bianchi identites:

$$0 = \begin{pmatrix} 0 & \Psi & 0 & {}^t\boldsymbol{\Phi} \\ 0 & 0 & 2\,\Psi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\,\boldsymbol{\Phi} & \boldsymbol{\Theta} \end{pmatrix} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \boldsymbol{\omega} \end{pmatrix}.$$

where $\Psi = d\psi$, $\boldsymbol{\Phi} = d\boldsymbol{\phi} + \boldsymbol{\theta}\wedge\boldsymbol{\phi}$, and $\boldsymbol{\Theta} = d\boldsymbol{\theta} + \boldsymbol{\theta}\wedge\boldsymbol{\theta}$.

By the second line of this system, $\Psi\wedge\omega_3 = 0$, while the first line implies that $\Psi\wedge\omega_2 \equiv 0 \mod \boldsymbol{\omega}$, so there must be functions p and $\mathbf{q}$, with values in $\mathbb{R}$ and $\mathbb{O}$ respectively, so that

$$\Psi = (p\,\omega_2 + \mathbf{q}\cdot\boldsymbol{\omega})\wedge\omega_3\,.$$

Substituting this into the first line of the system yields

$${}^t\big(\boldsymbol{\Phi} - \mathbf{q}\,\omega_2\wedge\omega_3\big)\wedge\boldsymbol{\omega} = 0,$$

so it follows that

$$\boldsymbol{\Phi} = \mathbf{q}\,\omega_2\wedge\omega_3 + \boldsymbol{\sigma}\wedge\boldsymbol{\omega}\,,$$

where $\boldsymbol{\sigma} = {}^t\boldsymbol{\sigma}$ is some 1-form with values in the symmetric part of $\mathfrak{gl}(\mathbb{O})$, which will be denoted $S^2(\mathbb{O})$ from now on. Substituting this last equation into the last line of the Bianchi identities, yields

$$2\,\boldsymbol{\sigma}\wedge\boldsymbol{\omega}\wedge\omega_3 + \boldsymbol{\Theta}\wedge\boldsymbol{\omega} = \mathbf{0}.$$

In particular, this implies that $\boldsymbol{\Theta}{\wedge}\boldsymbol{\omega} = \mathbf{0} \bmod \omega_3$, so that $\boldsymbol{\Theta} \equiv \mathbf{R}\bigl(\boldsymbol{\omega}{\wedge}\boldsymbol{\omega}\bigr) \bmod \omega_3$ where $\mathbf{R}$ is a function on B with values in $\mathcal{K}\bigl(\mathfrak{spin}(7)\bigr)$, which is the irreducible Spin(7) module of highest weight $(0,2,0)$ and of (real) dimension 168. (This uses the usual calculation of the curvature tensor of Spin(7)-manifolds.) Thus, set

$$\boldsymbol{\Theta} = \mathbf{R}\bigl(\boldsymbol{\omega}\wedge\boldsymbol{\omega}\bigr) + 2\,\boldsymbol{\alpha}\wedge\omega_3\,,$$

where $\boldsymbol{\alpha}$ is a 1-form with values in $\mathfrak{spin}(7)$ whose entries can be assumed, without loss of generality, to be linear combinations of ω_1, ω_2, and the components of $\boldsymbol{\omega}$. Substituting this last relation into the last line of the Bianchi identities now yields

$$2\,\boldsymbol{\sigma}\wedge\boldsymbol{\omega}\wedge\omega_3 + 2\,(\boldsymbol{\alpha}\wedge\omega_3)\wedge\boldsymbol{\omega} = \mathbf{0},$$

which is equivalent to the condition

$$\boldsymbol{\sigma}\wedge\boldsymbol{\omega} \equiv \boldsymbol{\alpha}\wedge\boldsymbol{\omega} \bmod \omega_3.$$

In particular, this implies that $\boldsymbol{\sigma} - \boldsymbol{\alpha} \equiv 0 \bmod \omega_3, \boldsymbol{\omega}$. Since $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$ take values in $S^2(\mathbb{O})$ and $\mathfrak{spin}(7)$ respectively, which are disjoint subspaces of $\mathfrak{gl}(\mathbb{O})$, it follows that $\boldsymbol{\sigma} \equiv \boldsymbol{\alpha} \equiv 0 \bmod \omega_3, \boldsymbol{\omega}$. In particular, neither ω_1 nor ω_2 appear in the expressions for $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$. Recall that, by definition, ω_3 does not appear in the expression for $\boldsymbol{\alpha}$, so $\boldsymbol{\alpha}$ must be a linear combination of the components of $\boldsymbol{\omega}$ alone. Now, from the above equation, it follows that

$$\boldsymbol{\sigma}\wedge\boldsymbol{\omega} = \boldsymbol{\alpha}\wedge\boldsymbol{\omega} + \mathbf{s}\,\omega_3\wedge\boldsymbol{\omega}$$

where $\mathbf{s}$ takes values in $S^2(\mathbb{O})$. Finally, the first line of the Bianchi identities show that ${}^t\boldsymbol{\omega}{\wedge}\boldsymbol{\alpha}{\wedge}\boldsymbol{\omega} = 0$, so it follows that $\boldsymbol{\alpha} = \mathbf{a}(\boldsymbol{\omega})$ where $\mathbf{a}$ is a function on B that takes values in a subspace of $\mathrm{Hom}\bigl(\mathbb{O}, \mathfrak{spin}(7)\bigr)$ that is of dimension $8\cdot 21 - 56 = 112$. By the usual weights and roots calculation, it follows that this subspace is irreducible, with highest weight $(0,1,1)$.

To summarize, the Bianchi identities show that the curvature of a torsion-free $\rho(H)$-structure B must have the form

$$\begin{aligned}
\boldsymbol{\Psi} &= (p\,\omega_2 + \mathbf{q}\cdot\boldsymbol{\omega})\wedge\omega_3\,,\\
\boldsymbol{\Phi} &= \mathbf{q}\,\omega_2\wedge\omega_3 + \mathbf{s}\,\omega_3\wedge\boldsymbol{\omega} + \mathbf{a}(\boldsymbol{\omega})\wedge\boldsymbol{\omega}\\
\boldsymbol{\Theta} &= \mathbf{R}\bigl(\boldsymbol{\omega}\wedge\boldsymbol{\omega}\bigr) + 2\,\mathbf{a}(\boldsymbol{\omega})\wedge\omega_3
\end{aligned}$$

where $\mathbf{R}$ takes values in $\mathcal{K}\bigl(\mathfrak{spin}(7)\bigr)$, the irreducible Spin(7)-representation of highest weight $(0,2,0)$ (of dimension 168), $\mathbf{a}$ takes values in the irreducible Spin(7)-representation of highest weight $(0,1,1)$ (of dimension 112), $\mathbf{s}$ takes values in $S^2(\mathbb{O})$ (the sum of a trivial representation with an irreducible one of highest weight $(0,0,2)$ and of dimension 35), $\mathbf{q}$ takes values in $\mathbb{O}$, and p takes values in $\mathbb{R}$. Thus, the curvature space $\mathcal{K}\bigl(\rho'(\mathfrak{h})\bigr)$ has dimension 325. By inspection, this curvature space passes Berger's first test (i.e., the generic element has the full $\rho'(\mathfrak{h})$ as its range). Thus, a structure with the full holonomy is not ruled out by this method.

3. Integrating the structure equations. To go further in the analysis, it will be useful to integrate the structure equations, at least locally. This will be done by a series of observations.

To begin, notice that, since $d\omega_3 = 0$, there exists, locally, a function x_3 on M so that $\omega_3 = dx_3$. This function is determined up to an additive constant, and can be defined on any simply connected open subset $U_0 \subset M$.

Since $d\omega_2 = -2\,\psi{\wedge}\omega_3 = -2\,\psi{\wedge}dx_3$, it follows that any point of U_0 has an open neighborhood $U_1 \subset U_0$ on which there exists a function x_2 for which $\omega_2{\wedge}\omega_3 = dx_2{\wedge}dx_3$. The function x_2 is determined up to the addition of an arbitrary function of x_3. In consequence, there exists a function r on $B_1 = \pi^{-1}(U_1)$ so that $\omega_2 = dx_2 - 2r\,dx_3$. It now follows from the structure equation for $d\omega_2$ that $\psi{\wedge}\omega_3 = dr{\wedge}dx_3$. Consequently, there is a function f on B_1 so that $\psi = dr + f\,dx_3$. Since $\mathbf{\Psi} = d\psi$ is π-basic, it follows that $df{\wedge}dx_3$ is well-defined on U_1. Consequently, f is well-defined on U_1 up to the addition of an arbitrary function of x_3.

Now, since

$$d\omega_1 = -\psi\wedge\omega_2 - {}^t\boldsymbol{\phi}\wedge\boldsymbol{\omega} = -(dr + f\,dx_3)\wedge(dx_2 - 2r\,dx_3) - {}^t\boldsymbol{\phi}\wedge\boldsymbol{\omega},$$

it follows that

$$d(\omega_1 + r\,dx_2 - r^2\,dx_3) = f\,dx_2\wedge dx_3 - {}^t\boldsymbol{\phi}\wedge\boldsymbol{\omega}.$$

The fact that the 2-form on the right hand side is closed, together with the fact that the system I of dimension 9 spanned by dx_3 and the components of $\boldsymbol{\omega}$ is integrable (which follows from the structure equations), implies that there are functions G and $\mathbf{F}$ on B so that

$$d(\omega_1 + r\,dx_2 - r^2\,dx_3) = d(G\,dx_3 - {}^t\mathbf{F}\,\boldsymbol{\omega}),$$

from which it follows that there is a function x_1 on B so that

$$\omega_1 = dx_1 - r\,dx_2 + r^2\,dx_3 + G\,dx_3 - {}^t\mathbf{F}\,\boldsymbol{\omega}\,.$$

The function x_1 is determined (once the choices of x_3 and x_2 are made) up to an additive function that is constant on the leaves of the system I, i.e., up to the addition of an (arbitrary) function of 9 variables. Expanding $d(G\,dx_3 - {}^t\mathbf{F}\,\boldsymbol{\omega}) = f\,dx_2{\wedge}dx_3 - {}^t\boldsymbol{\phi}{\wedge}\boldsymbol{\omega}$ via the structure equations and reducing modulo dx_3 yields

$${}^t\big(d\mathbf{F} + \boldsymbol{\theta}\,\mathbf{F}\big)\wedge\boldsymbol{\omega} \equiv {}^t\boldsymbol{\phi}\wedge\boldsymbol{\omega} \bmod dx_3\,.$$

so that there must exist functions $\mathbf{H}$ and $\mathbf{u} = {}^t\mathbf{u}$ so that

$$\boldsymbol{\phi} = d\mathbf{F} + \boldsymbol{\theta}\,\mathbf{F} + \mathbf{H}\,dx_3 + \mathbf{u}\,\boldsymbol{\omega}\,.$$

Substituting this back into the relation $d(G\,dx_3 - {}^t\mathbf{F}\,\boldsymbol{\omega}) = f\,dx_2{\wedge}dx_3 - {}^t\boldsymbol{\phi}{\wedge}\boldsymbol{\omega}$ yields

$$dG + 2\,{}^t\mathbf{F}\,d\mathbf{F} - {}^t\big(\mathbf{H} - 2\mathbf{u}\,\mathbf{F}\big)\,\boldsymbol{\omega} \equiv f\,dx_2 \bmod dx_3\,.$$

Setting $G = g - \mathbf{F}\cdot\mathbf{F}$ and $\mathbf{h} = \mathbf{H} - 2\mathbf{u}\,\mathbf{F}$, this becomes

$$dg \equiv f\,dx_2 + {}^t\mathbf{h}\,\boldsymbol{\omega} \mod dx_3\,,$$

with the formulae

$$\begin{aligned}\omega_1 &= dx_1 - r\,dx_2 + r^2\,dx_3 + (g{-}\mathbf{F}\cdot\mathbf{F})\,dx_3 - {}^t\mathbf{F}\,\boldsymbol{\omega}\,,\\ \boldsymbol{\phi} &= d\mathbf{F} + \boldsymbol{\theta}\,\mathbf{F} + (\mathbf{h} + 2\mathbf{u}\,\mathbf{F})\,dx_3 + \mathbf{u}\,\boldsymbol{\omega}\,.\end{aligned}$$

Now the final structure equation becomes

$$d\boldsymbol{\omega} = -2\big(d\mathbf{F} + \boldsymbol{\theta}\,\mathbf{F} + \mathbf{u}\,\boldsymbol{\omega}\big)\wedge dx_3 - \boldsymbol{\theta}\wedge\boldsymbol{\omega}$$

which can be rearranged to give

$$d\big(\boldsymbol{\omega} + 2\mathbf{F}\,dx_3\big) = -\big(\boldsymbol{\theta} - 2\mathbf{u}\,dx_3\big)\wedge\big(\boldsymbol{\omega} + 2\mathbf{F}\,dx_3\big)\,.$$

This suggests setting $\boldsymbol{\eta} = \boldsymbol{\omega} + 2\mathbf{F}\,dx_3$ and writing the formulae found so far as

$$\begin{aligned}\omega_1 &= dx_1 - r\,dx_2 + r^2\,dx_3 + (g{+}\mathbf{F}\cdot\mathbf{F})\,dx_3 - {}^t\mathbf{F}\,\boldsymbol{\eta}\,,\\ \omega_2 &= dx_2 - 2r\,dx_3\,,\\ \omega_3 &= dx_3\,,\\ \boldsymbol{\omega} &= -2\mathbf{F}\,dx_3 + \boldsymbol{\eta}\,,\\ \psi &= dr + f\,dx_3\,,\\ \boldsymbol{\phi} &= d\mathbf{F} + \boldsymbol{\theta}\,\mathbf{F} + \mathbf{h}\,dx_3 + \mathbf{u}\,\boldsymbol{\eta}\,,\\ dg &\equiv f\,dx_2 + {}^t\mathbf{h}\,\boldsymbol{\eta} \mod dx_3\,,\\ d\boldsymbol{\eta} &= -\big(\boldsymbol{\theta} - 2\mathbf{u}\,dx_3\big)\wedge\boldsymbol{\eta}\,.\end{aligned}$$

where, in these equations, $\boldsymbol{\theta}$ takes values in $\mathfrak{spin}(7)$ and $\mathbf{u} = {}^t\mathbf{u}$. Note that

$$-4\,\omega_1\,\omega_3 + {\omega_2}^2 + \boldsymbol{\omega}\cdot\boldsymbol{\omega} = -4\,dx_1\,dx_3 + {dx_2}^2 - 4g\,{dx_3}^2 + \boldsymbol{\eta}\cdot\boldsymbol{\eta}.$$

4. Interpreting the integration. I now want to describe how these formulae give a recipe for writing down all of the solutions to our problem.

By the last of the structure equations, the eight components of $\boldsymbol{\eta}$ describe an integrable system of rank 8 that is (locally) defined on the original 11-manifold. Let us restrict to a neighborhood where the leaf space of $\boldsymbol{\eta}$ is simple, i.e., is a smooth manifold K^8. The equation $d\boldsymbol{\eta} = -\big(\boldsymbol{\theta} - 2\mathbf{u}\,dx_3\big)\wedge\boldsymbol{\eta}$ shows that on $\mathbb{R}\times K^8$, with coordinate x_3 on the first factor, there is a $\{1\}\times\mathrm{Spin}(7)$-structure, which can be thought of as a 1-parameter family of torsion-free Spin(7)-structures on K^8 (the parameter is x_3, of course).

This 1-parameter family is not arbitrary because the matrix $\mathbf{u}$ is symmetric. This condition is equivalent to saying that if Φ is the canonical Spin(7)-invariant 4-form (depending on x_3, of course) then

$$\frac{\partial\Phi}{\partial x_3} = \lambda\,\Phi + \Upsilon$$

for some function λ on $\mathbb{R}\times K^8$ and Υ is an anti-self dual 4-form (via the x_3-dependent metric on the fibers of $\mathbb{R}\times K \to \mathbb{R}$, of course). It is not hard to see that this is 7 equations on the variation of torsion-free Spin(7)-structures and that, moreover, given any 1-parameter variation of torsion-free Spin(7)-structures, one can (locally) gauge this family by diffeomorphisms preserving the fibers of $\mathbb{R}\times K \to \mathbb{R}$ so that it satisfies these equations. (In fact, if K is compact and the cohomology class of Φ in $H^4(K,\mathbb{R})$ is independent of x_3 then this can be done globally.) Call such a variation *conformally anti-self dual*.

Now from the above calculations, this process can be reversed: One starts with any conformally anti-self dual variation of Spin(7)-structures on K^8, then on $\mathbb{R}^3\times K$ one forms the Lorentzian metric

$$ds^2 = -4\,dx_1\,dx_3 + {dx_2}^2 - 4g\,{dx_3}^2 + \boldsymbol{\eta}\cdot\boldsymbol{\eta}$$

where g is any function on $\mathbb{R}^3\times K$ that satisfies $\partial g/\partial x_1 = 0$ and $\boldsymbol{\eta}\cdot\boldsymbol{\eta}$ is the x_3-dependent metric associated to the variation of Spin(7)-structures. Then this Lorentzian metric has a parallel null spinor. For generic choice of the variation of Spin(7)-structures and the function g, this will yield a Lorentzian metric whose holonomy is the desired stabilizer group of dimension 30. This can be seen by combining the standard generality result for Spin(7)-metrics on 8-manifolds, which shows that for generic choices as above the curvature tensor has range equal to the full $\rho'(\mathfrak{h})$ at the generic point, with the Ambrose-Singer holonomy Theorem, which implies that such a metric will have its holonomy equal to the full group of dimension 30.

In particular, it follows that, up to diffeomorphism, the local solutions to this problem depend on one arbitrary function of 10 variables. It has to be remarked, though, that such a solution is not, in general, Ricci flat, in contrast to the case where a $(10,1)$ metric has a non-null parallel spinor field.

Note, by the way, that the 4-form Φ will not generally be closed, let alone parallel. However, the 5-form $dx_3{\wedge}\Phi$ will be closed and parallel. The other non-trivial parallel forms are the 1-form dx_3, the 2-form $dx_2{\wedge}dx_3$, and the 6-, 9-, and 10-forms that are the duals of these.