# EXAM 2

Math 219, 2022 Fall, Clark Bray.

Name: Solutions Section:______ Student ID:____________

## GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT.
CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

## WRITING RULES

Do not write anything near the staple – this may be cut off.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

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## DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

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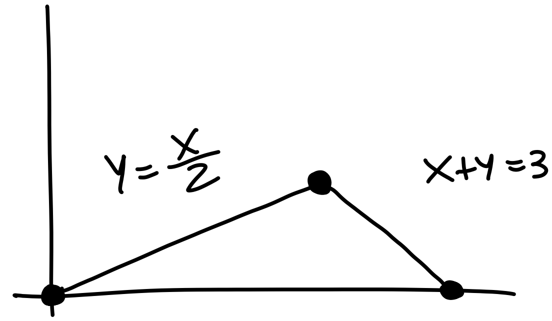
1. *(20 pts)* A wildlife preserve is in the shape of a triangle with vertices at $(0,0)$, $(2,1)$, and $(3,0)$. There is a river along the $y$-axis attracting the deer, causing the population density of deer in the preserve to be $\delta(x,y) = 100e^{-x}$. Compute the total number of deer in the preserve.




$$P = \iint_R dP = \iint_R \delta \, dA$$

$$= \int_0^1 \int_{2y}^{3-y} 100\, e^{-x} \, dx \, dy$$

$$= \int_0^1 -100 e^{-x} \Big]_{2y}^{3-y} dy = 100 \int_0^1 e^{-2y} - e^{y-3} \, dy$$

$$= 100 \left( -\tfrac{1}{2} e^{-2y} - e^{y-3} \right]_0^1 = 100 \left( \left( -\tfrac{3}{2} e^{-2} \right) - \left( -\tfrac{1}{2} - e^{-3} \right) \right)$$

$$= 100 \left( \tfrac{1}{2} + e^{-3} - \tfrac{3}{2} e^{-2} \right)$$

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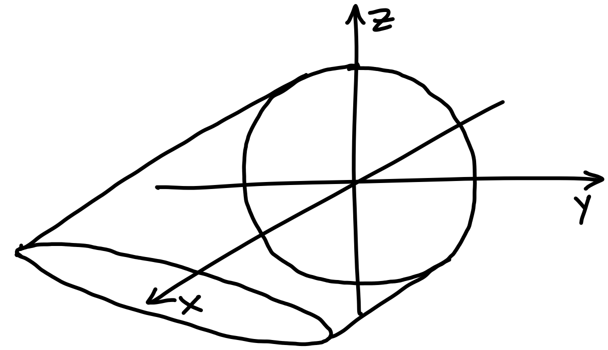
2. *(20 pts)*

(a) Set up an iterated integral to compute $\iiint_D x^2 e^{x^2 z^3} \sin y + y^3 \, dV$, where $D$ is the region bounded by $y^2 + z^2 = 4$, $x = z + 4$, and $x = 0$. (Do NOT evaluate this integral in this part of the question.)




$$\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{0}^{z+4} \left( x^2 e^{x^2 z^3} \sin y + y^3 \right) dx\, dz\, dy$$

(b) Compute the original integral from part (a), using any methods from this course.

$D$ is symmetric through the $xz$-plane, with $R(x,y,z) = (x,-y,z)$.

And the integrand $f$ is odd through this plane:

$$f(R(x,y,z)) = f(x,-y,z) = x^2 e^{x^2 z^3} \sin(-y) + (-y)^3$$
$$= -x^2 e^{x^2 z^3} \sin y - y^3$$
$$= -f(x,y,z).$$

So the integral is zero by symmetry.

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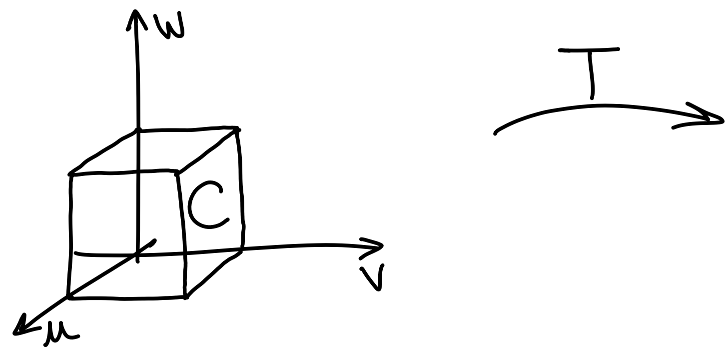
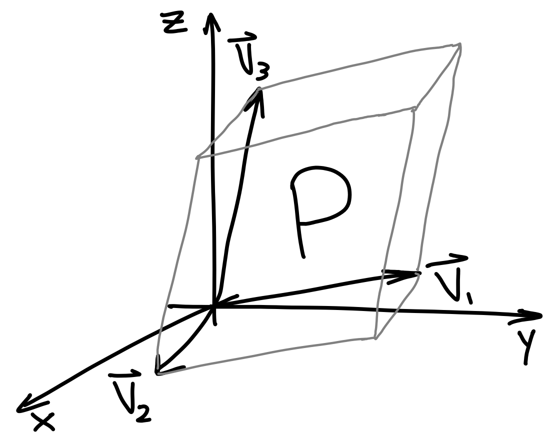
3. *(20 pts)* The parallelepiped $P$ has one vertex at the origin and edges defined by the vectors $(0,3,1)$, $(2,1,0)$, $(0,1,4)$. Compute $\iiint_P x\,dx\,dy\,dz$.




$$P = T(C), \text{ where } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = T\begin{pmatrix} u \\ v \\ w \end{pmatrix} = A\begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 2 & 0 \\ 3 & 1 & 1 \\ 1 & 0 & 4 \end{pmatrix}$$

$$\det(DT) = \det A = -2(3\cdot 4 - 1\cdot 1) = -22.$$

Then

$$\iiint_P x \, dx\,dy\,dz = \iiint_C x \, |\det DT| \, du\,dv\,dw$$

$$= \iiint_C (2v)(22)\, du\,dv\,dw$$

$$= 44 \int_0^1\int_0^1\int_0^1 v \, dv\,du\,dw$$

$$= 44 \int_0^1\int_0^1 \tfrac{1}{2} \, du\,dw \quad = 22$$

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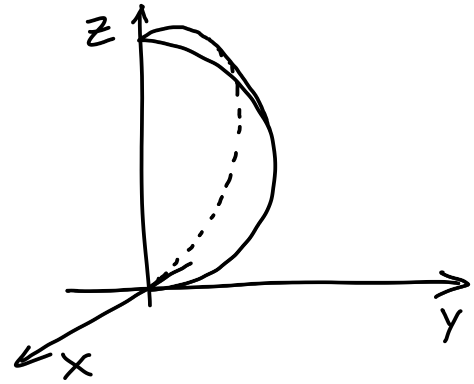
4. *(20 pts)* The solid $R$ is defined by $x^2 + y^2 + (z-2)^2 \leq 4$, $x \leq 0$, $y \geq 0$. Set up as a triple iterated integral in spherical coordinates (but do not evaluate!) the integral $\iiint_R xyz\, dV$.

$$x^2 + y^2 + z^2 - 4z + 4 = 4$$

$$\rho^2 - 4\rho\cos\phi = 0$$

$$\rho = 4\cos\phi$$




$$\iiint_R xyz\, dV$$

$$= \int_{\pi/2}^{\pi}\int_0^{\pi/2}\int_0^{4\cos\phi} (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\,(\rho^2\sin\phi)\, d\rho\, d\phi\, d\theta$$

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For each of the statements below (2 points each, 20 points total on this page), read and consider CAREFULLY, and then decide if the statement is true or false. Check the "T" box for true, or the "F" box for false. (If you need to change your answer, check the "C" box and then *clearly* write "true" or "false" next to the statement.)

5. (T☑ F☐ C☐) For the differentiable function $f : \mathbb{R}^n \to \mathbb{R}^1$, the directional derivative $D_{\vec{p}} f(\vec{q})$ (where $\vec{p}$ is a unit vector) can be computed by $\frac{d}{dt}\Big|_{t=0} f(\vec{q} + t\vec{p})$.
See the formulas for directional derivative on page 39 of the lecture notes for chapter 2.

6. (T☐ F☑ C☐) A directional derivative of a differentiable function $f$ at a point $\vec{a}$ is always orthogonal to the level set of $f$ at the point $\vec{a}$.
It is the gradient vector, not a (or any) directional derivative (which in fact is a scalar and thus does not have a direction), that is always orthogonal to level sets. See page 43 of the chapter 2 lecture notes.

7. (T☑ F☐ C☐) The direction of fastest increase (aka steepest ascent) for a differentiable function $f$ is the direction of the gradient of $f$ (assuming it is nonzero).
See the discussions of "direction of fastest increase" on page 41 of the chapter 2 lecture notes.

8. (T☐ F☑ C☐) If $F : \mathbb{R}^n \to \mathbb{R}^1$ is $C^1$ and we consider $F(x_1, \ldots, x_n) = c$ containing $\vec{a} \in \mathbb{R}^n$, then we can establish that $\frac{\partial x_k}{\partial x_m}(\vec{a})$ is defined on that level set by showing that $\frac{\partial F}{\partial x_m}(\vec{a}) \neq 0$.
We need to know that the partial w.r.t x_k is nonzero in order to know that x_k can even be viewed as a function of the other variables. Knowing only that the partial w.r.t. x_m is nonzero does not give us any indication as to whether x_k can be viewed as a function.

9. (T☐ F☑ C☐) If a curve is parametrized by arclength, then the speed increases as you move along that curve.
If a curve is parametrized by arc length, then the speed is constant and equal to 1, not increasing. See page 2 of the chapter 3 lecture notes.

10. (T☐ F☑ C☐) The flux of a continuous vector field $\vec{F}$ through a flat area $A$ is defined ONLY if $\vec{F}$ represents the velocity of a fluid. We saw on pages 5 and 6 of the chapter 3 lecture notes that when F is density*velocity the flux also has a natural interpretation, as dm/dt instead of dV/dt. Also, we have seen no discussion indicating any failure of definition of flux when F represents forces, (and in fact we will see later in the course that flux in that context has useful interpretations too).

11. (T☑ F☐ C☐) The divergence of $\vec{F}(x, y, z) = (x^3 - y^4, y^3 - z^4, z^3 - x^4)$ is nowhere negative.
The divergence is 3x^2 + 3y^2 + 3z^2, which is never negative.

12. (T☑ F☐ C☐) The curl of a differentiable vector field $\vec{F}$ on $\mathbb{R}^3$ is also a vector field on $\mathbb{R}^3$.
See the formula defining curl on page 9 of the chapter 3 lecture notes.

13. (T☐ F☑ C☐) The curl of a differentiable vector field $\vec{F}$ on $\mathbb{R}^3$ is a measure of the curvature of the flow lines for that field.
See the example in the middle of page 10 of the chapter 3 lecture notes, where the curl is nonzero even though the flow lines (all straight lines) all have no curvature at all.

14. (T☐ F☑ C☐) If the divergence of a differentiable vector field $\vec{F}$ on $\mathbb{R}^3$ is zero, then there must be a function $f$ for which $\nabla f = \vec{F}$.
It is curl, not divergence, that when zero implies that a vector field is a gradient of another vector field. See the sequence at the bottom of page 10 of the chapter 3 lecture notes, and the corresponding discussions in the lecture recordings.

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