EXAM 1

Math 219, 2023 Spring, Clark Bray.

Name: Solutions ____________ Section: ______ Student ID: ____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything near the staple – this may be cut off.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________

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1. *(20 pts)*

(a) Parametrize the curve C (in the xy-plane) with equation $x\sin^2(y^2) + x - y = 0$.

We have $x = \dfrac{y}{1+\sin^2(y^2)}$.

Let $y = t$. Then

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{t}{1+\sin^2(t^2)} \\ t \end{pmatrix}$$

(b) The curve D is the result of rotating C counterclockwise around the origin by angle $\pi/6$. Parametrize the curve D.

$$R = \begin{pmatrix} \cos\pi/6 & -\sin\pi/6 \\ \sin\pi/6 & \cos\pi/6 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

Applying to the points of C above, we get

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} \dfrac{t}{1+\sin^2(t^2)} \\ t \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{3}/2\,\dfrac{t}{1+\sin^2(t^2)} - 1/2\,t \\ 1/2\,\dfrac{t}{1+\sin^2(t^2)} \quad \sqrt{3}/2\,t \end{pmatrix}.$$

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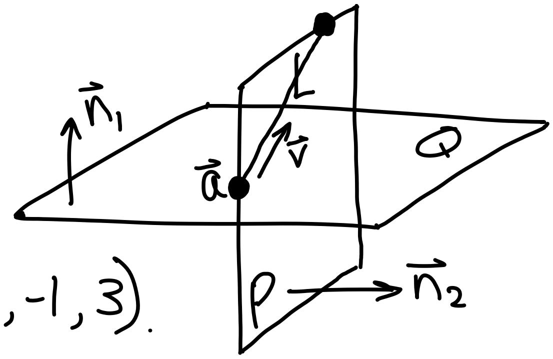
2. *(20 pts)*

(a) Find the equation of the plane P that contains the line $L = \{(t, 4-t, 1+2t)\}$ and that is perpendicular to the plane $Q = \{x + 2y + 3z = 0\}$.

$\vec{a} = (0,4,1) \quad \vec{v} = (1,-1,2) \quad \vec{n}_1 = (1,2,3)$



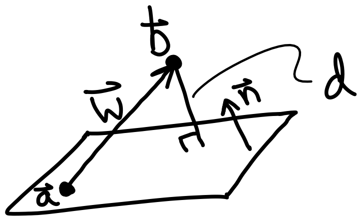
Need $\vec{n}_2 \perp \vec{v}, \vec{n}_1$, so choose

$$\vec{n}_2 = \vec{v} \times \vec{n}_1 = \det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & -1 & 2 \\ 1 & 2 & 3 \end{pmatrix} = (-7,-1,3).$$

Then P has equation $\vec{n}_2 \cdot \vec{x} = \vec{n}_2 \cdot \vec{a}$

$$-7x - 1y + 3z = -1$$

(b) Find the distance from $(1,0,1)$ to the plane $R = \{3x + 2y - 6z = 3\}$



$\vec{b} = (1,0,1), \quad \vec{n} = (3,2,-6).$

Choose $\vec{a} = (1,0,0)$ on R.

Then $\vec{w} = \vec{b} - \vec{a} = (0,0,1)$.

So $d = \left|\operatorname{comp}_{\vec{n}}(\vec{w})\right|$

$$= \left|\frac{\vec{n}\cdot\vec{w}}{\|\vec{n}\|}\right| = \left|\frac{(3,2,-6)\cdot(0,0,1)}{7}\right|$$

$$= \frac{6}{7}$$

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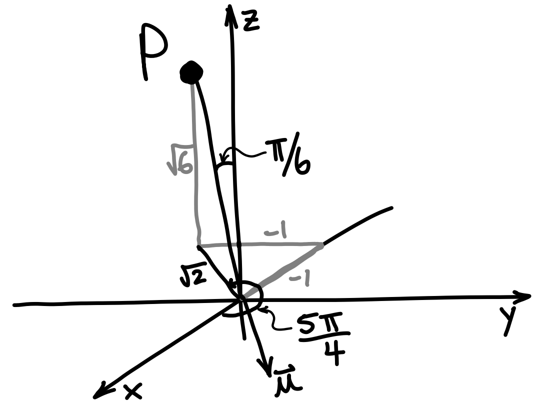
3. *(20 pts)*

(a) The point P has rectangular coordinates $(-1, -1, \sqrt{6})$. Find the unique set of spherical coordinates for P with $\theta \in [0, \pi]$ and $\phi \in [0, \pi]$.

Direction $\vec{u}$ is described by

$$\theta = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$$

$$\phi = -\frac{\pi}{6} + \pi = \frac{5\pi}{6}$$



This is opposite the direction of P. So $\rho = -2\sqrt{2}$.

(b) Find the polar equation for the circle of radius 3 centered at $(0, 3)$.

$$x^2 + (y-3)^2 = 3^2$$

$$x^2 + y^2 - 6y + 9 = 9$$

$$r^2 - 6r\sin\theta = 0$$

$$r = 6\sin\theta$$

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4. *(20 pts)* The surface S has equation $x^2 - y^2 + z = 1$.

(a) Find the function $f : \mathbb{R}^a \to \mathbb{R}^b$ whose graph is S, and identify a and b.

$z = 1 - x^2 + y^2$, is the graph $z = f(x,y)$ of

$f : \mathbb{R}^2 \to \mathbb{R}^1$ defined by $f(x,y) = 1 - x^2 + y^2$.

(b) Find a function $g : \mathbb{R}^c \to \mathbb{R}^d$ for which S is a level set, and identify c and d.

$x^2 - y^2 + z = 1$ is the level set $g^{-1}(1)$ of

$g : \mathbb{R}^3 \to \mathbb{R}^1$ defined by $g(x,y,z) = x^2 - y^2 + z$.

(c) Find the equation for the surface M that results from stretching S by a factor of 3 in the y-direction.

$$\underbrace{x^2 - y^2 + z = 1}_{S} \quad \xrightarrow[\text{"}y\text{"} \mapsto \text{"}\frac{y}{3}\text{"}]{\text{stretch}} \quad \underbrace{x^2 - \left(\frac{y}{3}\right)^2 + z = 1}_{M}$$

(d) Find the equation for the surface N that results from translating M by a distance 2 in the positive y-direction.

$$\underbrace{x^2 - \left(\frac{y}{3}\right)^2 + z = 1}_{M} \quad \xrightarrow[\text{"}y\text{"} \mapsto \text{"}y-2\text{"}]{\text{translate}} \quad \underbrace{x^2 - \left(\frac{y-2}{3}\right)^2 + z = 1}_{N}$$

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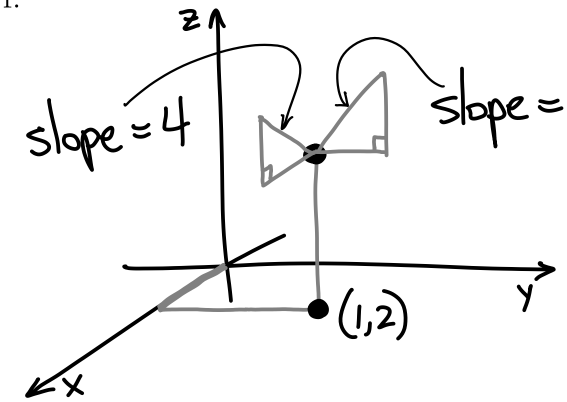
5. *(20 pts)*

(a) The graph $z = f(x, y)$ includes the point $P = (1, 2, 3)$. At P, the slope of the tangent line in the x-direction is 4 and the slope of the tangent line in the y-direction is 5. Use this information to find a linear approximation of f.

$$\vec{a} = (1,2), \quad f(\vec{a}) = 3$$

$$\frac{\partial f}{\partial x}(\vec{a}) = 4, \quad \frac{\partial f}{\partial y}(\vec{a}) = 5$$



$$L(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

$$= 3 + \begin{pmatrix} 4 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y-2 \end{pmatrix}$$

(b) The velocity of a parametric curve is $\vec{v}(t) = (t, e^t, \sin t)$, and $\vec{x}(0) = (1, 2, 3)$. Find an expression giving the position function $\vec{x}(t)$.

$$\vec{x}(t) = \int \vec{v}(t)\, dt = \int \begin{pmatrix} t \\ e^t \\ \sin t \end{pmatrix} dt = \begin{pmatrix} \frac{1}{2}t^2 \\ e^t \\ -\cos t \end{pmatrix} + \vec{c}$$

$$@\, t=0: \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \vec{c} \quad \Longrightarrow \quad \vec{c} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$$\text{So} \quad \vec{x}(t) = \begin{pmatrix} \frac{1}{2}t^2 \\ e^t \\ -\cos t \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

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