EXAM 2

Math 219, 2023 Fall, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not remove the staple, tear pages out of the staple, or tamper with the exam packet in any way. Do not write anything near the staple – this may be cut off.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

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1. *(20 pts)* We consider here the function $f : \mathbb{R}^2 \to \mathbb{R}^1$ given by $f(x,y) = x^3y - y^2$, the point $\vec{a} = (1,1)$, and the vector $\vec{v} = (4,5)$.

(a) At $\vec{a}$, find the unit vector that points in the direction of fastest increase of the function f.

$$\nabla f = (3x^2y, x^3 - 2y)$$

$$\nabla f(\vec{a}) = (3,-1)$$

$$\vec{u} = \frac{\nabla f(\vec{a})}{\|\nabla f(\vec{a})\|} = \frac{(3,-1)}{\sqrt{10}}$$

(b) At $\vec{a}$, what is the directional derivative of f in the direction of $\vec{v}$?

$$\text{Direction} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{(4,5)}{\sqrt{41}} = \vec{w}$$

$$D_{\vec{w}}f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{w} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \end{pmatrix} / \sqrt{41} = \frac{7}{\sqrt{41}} = \frac{df}{ds}$$

(c) If $\vec{x}$ is at $\vec{a}$ moving with velocity $\vec{v}$, what is the rate of change of $f(\vec{x})$ with respect to time?

$$\frac{df}{dt} = \frac{df}{ds}\frac{ds}{dt} = \left(D_{\vec{w}}f(\vec{a})\right)\|\vec{v}\| = \frac{7}{\sqrt{41}}\sqrt{41} = 7$$

$$\left(\text{Alt: } \frac{df}{dt} = \nabla f(\vec{a}) \cdot \vec{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \end{pmatrix} = 7\right)$$

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2. *(20 pts)*

(a) Find an antigradient for the vector field $\vec{F}(x,y) = (3, 2y)$.

$$\vec{F} = (3,2y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \nabla f \quad \Rightarrow \frac{\partial f}{\partial x} = 3, \quad \frac{\partial f}{\partial y} = 2y$$

These suggest terms $3x$ and y^2.

$$\nabla(3x+y^2) = (3,2y) \quad \checkmark$$

So $f(x,y) = 3x+y^2$ is an antigradient.

(b) The vector field $\vec{G}(x,y,z) = (3,1,2) = \delta\vec{v}$ represents the flow of a fluid with density δ and velocity $\vec{v}$. The surface S is a rectangle in the plane $2x+3y-4z$ with area 8. Compute the flow rate dm/dt of this fluid through the surface S.

$$\frac{dm}{dt} = \Phi = (\vec{G}\cdot\vec{n})A$$

unit normal vector is $\vec{n} = \dfrac{(2,3,-4)}{\sqrt{29}}$.

Then $\Phi = \left(\begin{pmatrix}3\\1\\2\end{pmatrix}\cdot\begin{pmatrix}2\\3\\-4\end{pmatrix}\Big/\sqrt{29}\right)8 = \dfrac{8}{\sqrt{29}}$.

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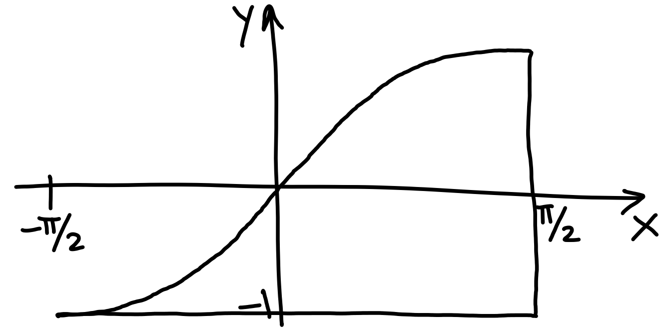
3. *(20 pts)* The region R in the xy-plane is bounded by $y = -1$, $x = \pi/2$, and the part of $y = \sin x$ with $-\pi/2 \le x \le \pi/2$. Mass is distributed across R with density $\delta(x, y) = 2 + 2y$. Compute the total mass in R.



$$m = \iint_R dm$$

$$= \iint_R \delta \, dA$$

$$= \int_{-\pi/2}^{\pi/2} \int_{-1}^{\sin x} (2+2y) \, dy \, dx$$

$$= \int_{-\pi/2}^{\pi/2} \left(2y + y^2\right]_{y=-1}^{y=\sin x} dx$$

$$= \int_{-\pi/2}^{\pi/2} \left(2\sin x + \sin^2 x\right) - (-1) \, dx$$

$$= \int_{-\pi/2}^{\pi/2} 2\sin x + \frac{1-\cos(2x)}{2} + 1 \, dx$$

$$= -2\cos x - \frac{1}{4}\sin(2x) + \frac{3}{2}x \Big]_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}$$

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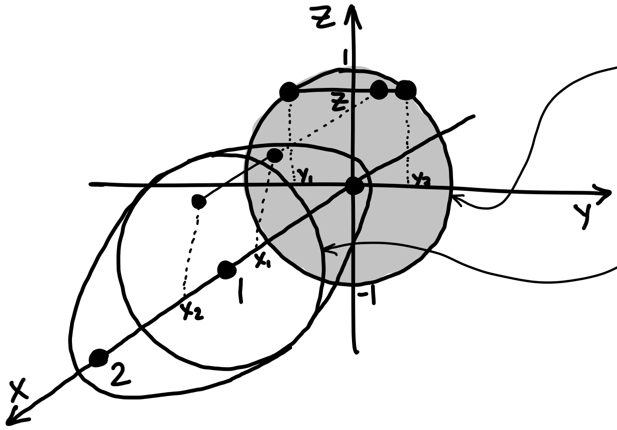
4. *(20 pts)* In this question we consider the function $f(x, y, z) = \cos^8(x) - \cos^8(2 - x)$, the solid region D that is bounded by the surfaces $x = y^2 + z^2$ and $x = 2 - y^2 - z^2$, and the triple integral $I = \iiint_D f(x, y, z)\, dV$.

(a) Write I as an iterated integral in rectangular coordinates. (Do NOT evaluate the iterated integral here.)



yz-projection determined by

$x_1 = x_2$

$y^2 + z^2 = 2 - y^2 - z^2$

$y^2 + z^2 = 1$

$$\iiint_D f\, dV = \int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{y^2+z^2}^{2-y^2-z^2} \left(\cos^8(x) - \cos^8(2-x)\right) dx\, dy\, dz$$

(b) Compute the value of I using a (valid and complete!) symmetry argument.

D is symmetric through $x=1$, $R(x,y,z) = (2-x, y, z)$.

f is odd through this plane:

$$\begin{aligned} f(R(x,y,z)) &= f(2-x, y, z) \\ &= \cos^8(2-x) - \cos^8(2-(2-x)) \\ &= \cos^8(2-x) - \cos^8(x) \\ &= -\left(\cos^8(x) - \cos^8(2-x)\right) = -f(x,y,z) \end{aligned}$$

So this integral is zero by symmetry.

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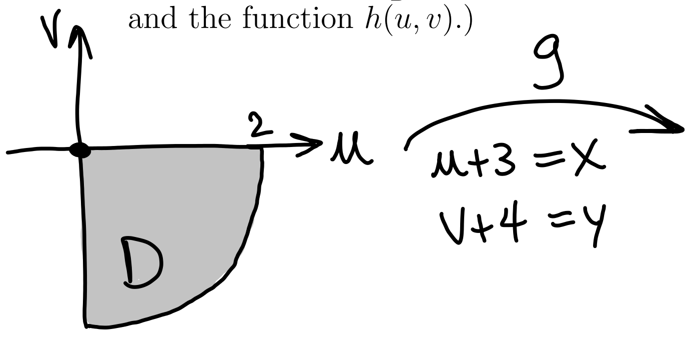
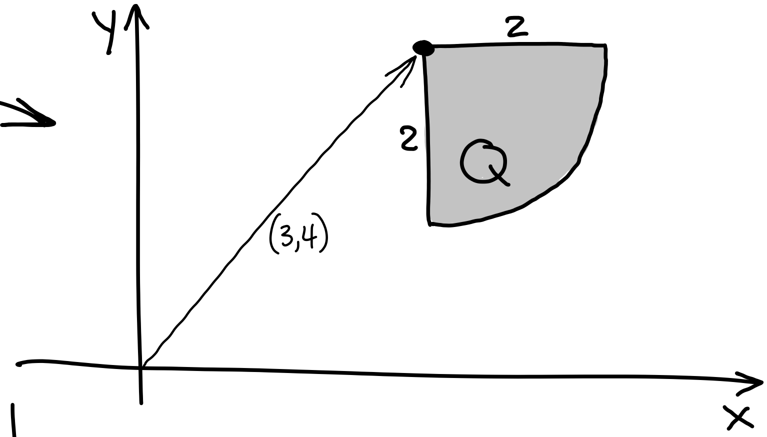
5. *(20 pts)* The region Q in the xy-plane is the quarter disk defined by $(x-3)^2+(y-4)^2 \leq 4$, $x \geq 3$, and $y \leq 4$; and we have $f(x,y) = x$. In this question we will be interested in the value of $\iint_Q f(x,y)\,dx\,dy$.

(a) Use the change of variables function $(x,y) = g(u,v) = (u,v)+(3,4)$ to rewrite $\iint_Q f(x,y)\,dx\,dy$ as a new integral $\iint_D h(u,v)\,du\,dv$. (That is, identify explicitly the region D in the uv-plane, and the function $h(u,v)$.)





$$\det Dg = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$$\iint_D (u+3)(1)\,du\,dv \longleftarrow \iint_Q (x)\,dx\,dy$$

D is the pictured quarter disk, and $h(u,v) = u+3$.

(b) Compute the value of the above integral using any methods from this course.

We use polar coordinates in the uv-plane ($u = r\cos\theta$, $v = r\sin\theta$).

$$\iint_D h(u,v)\,du\,dv = \int_{-\pi/2}^{0}\int_0^2 (u+3)(r\,dr\,d\theta)$$

$$= \int_{-\pi/2}^{0}\int_0^2 r^2\cos\theta + 3r\,dr\,d\theta = \int_{-\pi/2}^{0} \left(\frac{1}{3}r^3\cos\theta + \frac{3}{2}r^2\right]_{r=0}^{r=2} d\theta$$

$$= \int_{-\pi/2}^{0} \frac{8}{3}\cos\theta + 6\,d\theta = \left(\frac{8}{3}\sin\theta + 6\theta\right]_{-\pi/2}^{0} = \frac{8}{3} + 3\pi$$

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